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COURSES AND LECTURES No. 93

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COURSE HELD AT THE DEPARTMENT
OF GENERAL MECHANICS
OCTOBER 1972

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P R E F A C E

When discussing the mechanics of soil, rocks and loose media the models of the so called mechanics of continuous media are in general use. This model assumes the invariant of the contact relations between the elements of the media. In case of the above media being in motion the invariant relation of the contacts is not maintained. Contacts between these elements change, the ordered relation is not maintained, and the elements intermingle. The motion of the medium is characterized by the mass character of random changes in contact relations and consequently by random displacement of the medium elements.

The movement of such a collection of elements depends on their mechanical properties only in a small degree, being mainly dependent on their spatial structure. Since the interaction of the elements has a mass and random character, the summary effect of displacements of elements is defined by random laws in agreement with the central limiting theorems.

These heuristic considerations suggest the idea of describing the displacements of a loose medium on the basis of a model different from the model of a model different from the model of a continuous medium.

That model may be regarded as a system of integral equations which are generalizations of the Smoluchowski equation describing the stochastic processes of the Markov type. In particular, from this system a parabolic system of differential equations, defining the mean values of displacement components of a loose medium, can be obtained.

Solutions for a number of cases of boundary conditions of this system have been given. The results have been compared with the displacement measurements obtained in experiments carried out in a loose medium in which the corresponding boundary conditions have been realized.

J. Litwiniszyn

Udine, October 1971

STOCHASTIC METHODS IN MECHANICS OF GRANULAR BODIES

The mechanical phenomena in so called continuous media explained by means of a model based on the concept of continuous include phenomena for the explanation of which a continuous model is inadequate. In some cases we may feel that the mathematical model by means of which we describe the phenomenon is continuous, whereas the actual phenomenon described by the model is not continuous. The concept of noncontinuity seems to be inherent in the world of events and unavoidable.

The opposition of these two types of models of media, based on the model of continuum and model of a discrete medium, is known from the beginning of the history of mechanics.

Trials of reconciling these two opposed points of view involve basic considerations on the set theory and evolution of the concept of continuum. However, the unification of these two points of view continues to be an open problem.

The procedure used by Lagrange to derive the equation of vibrating strings is a good example of the trials of unifying these two points of view. Lagrange considered an arranged collection of N points, the motion of each of which is described by a function differentiable according to time. This leads to a system of difference - differential equations, to which the limiting transition for $N \rightarrow \infty$ applies. Such a function requires appropriate regularity of the findings describing the material coordinates of the points of the medium.

The assumption of regularity of these functions imposes limits to the possibility of motion of the system. These limits depend on the assumption of a contact relation between the elements of the system, and consequently arrangement of the elements, the relation being an invariant during motion of the system.

The limitation following from such an assumption in many cases leads to qualitative discrepancies between the representation by the mathematical model and reality.

The limited class of admissible motions of the medium obviously does not apply to the phenomena of motion of rarefied gases. The phenomena of turbulent flow is another example.

As is known, L.F. Richardson expressed doubts concerning the term wind velocity, i.e. whether the function describing the coordinates of flowing elements of the medium are differentiable according to time. In the case of Brownian movements, the measure, in the sense used by Wiener, of the set of differentiable functions describing the movements of the diffusing particles is equal to zero. This means that nearly every trajectory of particles exhibiting Brownian movements is undifferentiable.

Invariance of contact relations is not maintained in the flow of fluids in porous media. On the whole, granular bodies in motion do not fulfil this relation. Two grains of gran ular medium lying in contact may separate after a brief period. In that case, the condition of topologic transformation is not fulfilled.

The phenomena described above characterize a geo metric property of the collection of elements forming the medium, namely contact relation of the elements. The continuity of the medium is characterized by this relation. The medium is continuous if it cannot be divided into noncontacting parts.

Movement of a continuous medium is described by a group of topologic trasformations with unchanging contact relations. In other words, during movement no new contacts are formed, and existing contacts are not destroyed. This phenomenon may be described as follows: Let a be the Lagrangian coordinate of the medium. Movement of the medium with reference to an immobile system of coordinates $\{x\}$ is described by the relation $x = f(a,t)$ where is time, and $a = f(a,0)$ is fulfilled.

The medium fulfils the condition of continuity during motion if for each value of the continuous function $\varepsilon = \varepsilon(t) \geqslant 0$ there exists a number $\delta \geqslant 0$ so that the condition $\rho(a,b) < \delta$ is fulfilled only for each point b: then $\rho[f(a,t), f(b,t)] < \varepsilon(t)$, where $\rho = \rho(\alpha,\beta)$ is the distance between points α and β. In other words, if the distance $\rho[(a,0),f(b,0)] = \rho(a,b)$ is not "very large", then $\rho[f(a,t),f(b,t)]$ also is "not very large".

Is this condition fulfilled by all media in motion? It can easily be demonstrated that it is not. The condition is not fulfilled by granular media, in which the contact relation of the grains is changeable. As an example let us take a granular medium consisting of a collection of sand grains. During motion of the medium, the grains do not maintain this relation invariably. Two grains lying in contact at a given moment, may lose that contact.

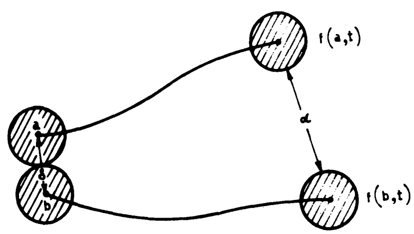

Fig. 1

This situation is illustrated in Fig. 1. However small δ is, for $\varrho(a,b) < \delta$ there is $\varrho[f(a,t), f(b,t)] \geqslant \alpha$ hence not "every" value of $\varepsilon(t)$ can be taken, because $\varepsilon(t) \geqslant \alpha$ must be fulfilled.

The last examples indicate that the condition of continuity is not maintained. In this case, application of the methods of mechanics of continuous media is inadequate. Neverthe less, these methods are widely used, e.g. in soil mechanics, or in classic mechanics of granular media. For the interpretation of this category of phenomena it is reasonable to use a different model, namely one in which the medium is regarded as a collection of discrete elements, the contact relations of which are not maintained during motion. Contacts between these elements change, the ordered relation is not maintained, and the elements intermingle. We have here a phenomenon similar to that which occurs in fluid or gas particles flowing turbulently, or performing Brownian motion.

However, the analogy between the motion of the elements of a granular medium with these phenomena is not complete. During the motion of granular media the freedom of movement of the elements is limited compared, for instance, with molecules of a rarefied gas, which has greater freedom.

2. Heuristic models based on the concept of random walk.

Motion of a granular medium is characterized by the mass character of the random changes in contact relations, and consequently random displacement of grains.

Hence, it is reasonable to regard the motion of a mass of a granular medium as random process. [1]

MODEL I

For preliminary heuristic considerations on the motion of a granular medium as a random process, let us imagine a system of cages illustrated in Fig. 2. Each cage contains a ball subjected to the force of gravity. Let the system of cages fulfil the condition that removal of a ball from the horizontal

in the second layer stratum to take its place, assuming equal probability of both events, i. e. 1/2.

As a result of the displacement of the ball from cage a_2 to a_1 , cage a_2 will be occupied by a ball from cage a_3

or b_3 from layer III. Similarly, removal of the ball from cage b_2 will cause its place to be taken by a ball from cage b_3 or c_3 .

Removal of a ball from cage a_1 empties one of the cages a_3 , b_3 or c_3 , the probability of these events being 1/4, 2/4 and 1/4 respectively. The distribution of the probabilities of these events in cages a_4 , b_4 , c_4 and d_4 in layer IV will be 1/8, 3/8 and 1/8.

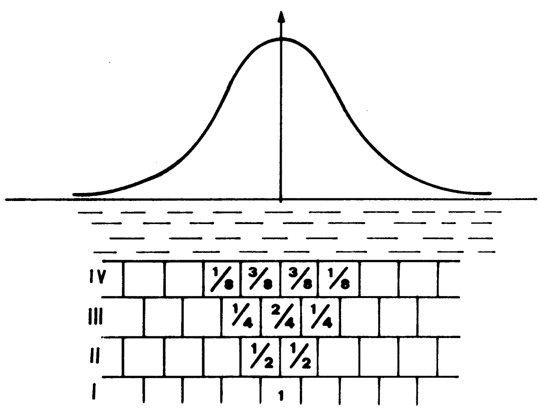

Fig. 2

The distribution of probabilities is illustrated in Fig. 2. Instead of one, a larger number of balls is removed from cage a_1 , then the cages in the highest layers which formerly contained balls will be emptied.

The boundary of the emptied cages forms a stepwise line. Given a sufficiently dense network of cages, removal of a sufficient number of balls from cage a_1 will give a stepwise line approaching Gauss's K curve, symmetrical with respect to a perpendicular straight line passing through the centre of cage a_1 .

The result described above, consisting in the formation of a contour of emptied cages in the upper layers of the system was predicted exclusively on the basis of elementary probability calculus, similarly to the prediction of the frequency of a coin cast into the air falling heads or tails up, or of drawing a given playing card from a deck.

In these predictions it is not the individual properties of the balls that count, but the structure of their collection.

Although the procedure outlined above does not permit description of the fate of individual balls, (given that a sufficiently large number of balls has been removed from the cage a_1) it does make it possible to predict some of the properties of the collection of balls (e.g. the law of distribution of the emptied cages).

The so called "limit theorems" of the theory of probability allow us to prognosticate. The practical importance of these theorems is that they show that mass random phenomena are governed by strict, not random, regularities. In other words, mass random events lead to a certain degree of regularity.

The choice of statistical methods as opposed to deterministic methods may be regarded either as an attempt to avoid conceptual and analytical difficulties, or as a desire to describe reality more accurately.

Regardless of the approach, however, the mathematical implications are the same.