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Introduction

Parabolic equations are one of the types of partial differential equations which have important applications to the natural sciences – e.g. in biomathematics or chemical combustion theory – and have long since enjoyed great popularity among researchers in pure and applied mathematics. Within this class of equations, periodic equations seem to be of particular interest since they can take into account seasonal fluctuations occurring in the phenomena they are modelling. The interest in this kind of problems is reflected, for instance, by the recent publication of the monograph [**67**]. Concrete examples are provided by periodic Volterra-Lotka population models with diffusion or the periodic Fisher equation of population genetics (see e.g. [**67**]).

A very simple example of a one-dimensional periodic-parabolic equation is the following initial-boundary value problem:

$$
(1) \qquad \begin{cases} \partial_t u(t,x) - k(t,x)\partial_x^2 u(t,x) = f\big(t, u(t,x)\big) & \text{for } (t,x) \in (0,\infty) \times (0,1) \\ u(t,0) = u(t,1) = 0 & \text{for } t \in (0,\infty) \\ u(0,x) = u_0(x) & \text{for } x \in [0,1] \end{cases}
$$

where $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k\colon \mathbb{R} \times [0,1] \to \mathbb{R}$, are smooth functions, T-periodic in the first argument, k additionally being strictly positive on $\mathbb{R} \times [0,1]$. The *initial value* u_0 is a given function from the unit interval into $\mathbb{R}$.

A *local classical solution* is a function

$$
u \in C\big([0,\varepsilon) \times [0,1], \mathbb{R}\big) \cap C^{1,2}\big((0,\varepsilon) \times [0,1], \mathbb{R}\big),
$$

for some $\varepsilon > 0$, which satisfies (1). The solution is called *global* if we can choose $\varepsilon = \infty$. A *T-periodic solution* is a global solution which is T-periodic in $t \in \mathbb{R}_+$.

Considering the fact that such an equation stems from the desire to model a real-world situation, the following requirements may sound reasonable:

1) Existence of unique solutions for a large class of initial values.
2) Continuous dependence of the solution on the initial data.
3) Globality of the solution.

4) Positivity of the solution whenever the initial value is positive.

The next step after establishing the above properties, is the study of qualitative behaviour of the model:

5) Do T-periodic solutions exist? (especially positive ones).
6) What are the stability and attractivity properties of such solutions? (ideally something like: all solutions having positive initial values converge towards a positive T-periodic solution as time approaches infinity).

For a long time one of the most successful methods for attacking this kind of problems has been the abstract formulation of (1) as an 'ordinary' differential equation in a suitable Banach space of functions. In order to convey the basic idea we consider equation (1), dispensing with the periodicity assumptions:

Let $u\colon [0,\varepsilon)\times[0,1]\to\mathbb{R}$ be a classical solution of (1), and set for each $t\in[0,\varepsilon)$: $v(t):=u(t,\cdot)\in C^2\big([0,1],\mathbb{R}\big)$. Then v is a continuously differentiable function from $(0,\varepsilon)$ into $C\big([0,1],\mathbb{R}\big)$. Setting:

$X_0 := \{u\in C\big([0,1],\mathbb{R}\big);\ u(0)=u(1)=0\}$ equipped with the supremum norm
$D(A(t)) := X_1 := \big\{w\in C^2\big([0,1],\mathbb{R}\big);\ w(0)=w(1)=0\big\}$,
$A(t)w := -k(t,\cdot)\partial_x^2 w$ for $w\in X_1$,
and $F(t,w) := f\big(t,w(\cdot)\big)$ for $w\in X_0$,

we see that v satisfies the *abstract initial value problem*:

$$\text{(2)}\qquad \begin{cases} \partial_t v(t)+A(t)v(t)=F\big(t,v(t)\big) & \text{for } t>0\\ v(0)=u_0, & \end{cases}$$

on the Banach space X_0.

The fundamental properties of the operator family $\big(A(t)\big)_{0\le t\le T}$ can be summarized by saying: for every $t\in[0,T]$, the operator $-A(t)$ is the generator of a strongly continuous analytic semigroup of bounded operators on the Banach space X_0. Furthermore, the nonlinearity F is a smooth function from X_0 into itself. For this kind of equation one can develop a theory which in many ways resembles the theory for ordinary differential equations.

It should be emphasized that the choice of the underlying Banach space, in our case $X_0=C\big([0,1],\mathbb{R}\big)$, is by no means unique. For example, one could equally well choose to work in $X_0=L_p\big((0,1)\big)$, $p\ge 1$. The choice of the Banach space should be adjusted to the particular pecularities of the problem under consideration.

The strategy, then, is to prove theorems for the abstract equation, and, by means of regularity results, translate them back into the context of classical solutions of (1).

One of the difficulties, when dealing with semilinear equations, which is not exemplified by (1), is the fact that the nonlinear term F may not have the degree of regularity needed for the abstract existence theory or, worse, may not even be defined as a function on the whole of X_0. This would be the case, for instance, if we allow the nonlinearity f in (1) to depend on $\partial_x u$. In this situation one would have to consider F as a function from $C^1\big([0,1],\mathbb{R}\big)$ into $C\big([0,1],\mathbb{R}\big)$. In general it might be necessary to view F as a function from Z into X_0, where Z is a Banach space lying between X_1 and X_0. We are thus confronted with the following question: which spaces lying between X_1 and X_0 provide the appropriate setting for our problem?

There are basically two approaches known which furnish suitable intermediate spaces:

1) Fractional power spaces associated to the operator A(0), and
2) Interpolation spaces.

Fractional power spaces are by far better known than interpolation spaces. This is the type of space used in the books [**66**], [**100**] and [**67**], to name just a few. While interpolation theory is certainly familiar to specialists working in existence theory, it is virtually unknown to mathematicians devoted to the qualitative theory of partial differential equations. This is in some respects a regrettable situation, as interpolation theory makes it possible to develop a much more elegant theory of semilinear equations and is capable of extension to problems having a much greater degree of generality (e.g. initial-boundary value problems where the boundary operator also depends on time and problems involving nonlinear boundary conditions). Another advantage is that while the definition of the fractional power spaces depends heavily on the operator $A(0)$, the corresponding interpolation spaces depend (up to equivalent norms) only on the norm isomorphic class of X_1 – equipped with the graph norm induced by $A(0)$ – and not on any other properties of $A(0)$.

By now the reader may have guessed what the intention of these notes is. We have set ourselves the task of giving a hopefully clear and essentially self-contained account of the theory for abstract evolution equations of the type (2), when the family $\big(A(t)\big)_{0\le t\le T}$ consists of operators having domains of definition, $D\big(A(t)\big)$, independent of $t \in [0,T]$, and the nonlinearity is defined on an interpolation space. Although these results are by no means restricted to periodic evolution equations we devote quite a deal of space to this type of problem. In order to keep the length of this volume within reasonable bounds, we have chosen to provide the basis for the qualitative analysis of parabolic equations rather than to actually carry out this analysis in concrete instances. In order to compensate this ommision we have tried to supply references where specific equations are considered.

In Section 0 we have collected a few basic functional analytic facts for easy reference.

We have also seized the opportunity to introduce some general notation which shall be used throughout the book.

In Chapter I we develop the fundamentals of the linear theory. After a brief review of the theory of analytic C_0-semigroups we introduce the notion of the evolution operator associated to a linear nonautonomous evolution equation and prove an existence theorem for solutions of such equations. The proof of the existence of the evolution operator is only sketched since it is very technical and easy to find in the existing literature. A quick introduction to interpolation theory from the user's perspective is given and – in order to convince the reader that interpolation spaces actually do exist – the most frequently used interpolation methods are described. Section 5 constitutes the core of the whole theory. There we prove the basic estimates for the evolution operator which will enable us to treat semilinear problems in Chapter IV.

Chapter II deals with linear periodic equations. We start by introducing the period-map associated to a linear periodic evolution equation and proceed to give some estimates for it which are related to the stability properties of the zero solution of the homogeneous equation. We also spend some time on proving estimates for the period-map involving spectral decompositions. They enable a more differentiated analysis of the asymptotics of solutions of the homogeneous equation. Finally, a theorem of 'Floquet-type' is proved which allows, for instance, to decouple the unstable part of a linear evolution equation.

Chapter III is a collection of results which are either of a very technical nature (and are therefore condemned to isolation), or are of a somewhat more specialized nature (and thus also condemned to isolation). It begins with the description of a general technique for solving Volterra integral equations. This is the technique with which the evolution operator is actually constructed. We prove that an evolution equation with unbounded principal part $A(t)$ may be approximated (in an appropriate sense) by a sequence of evolution equations whose principal parts are the Yosida-approximations of $A(t)$, and hence bounded. The question of how the evolution operator depends on a parameter is also addressed. Maximum principles for second order parabolic equations are described. They imply that the evolution operator associated with the abstract formulation of such problems is a positive irreducible operator. The last section in this chapter deals with superconvexity. This rather odd looking concept turns out to be quite useful when studying the linear stability properties of periodic-solutions to semilinear parabolic equations.

In Chapter IV we start the investigation of semilinear evolution equations. The concept of a mild solutions of (1) is defined and it is shown that – with an additional regularity condition on the nonlinearity – every mild solution is also a classical solution. Of course, we also give theorems on the existence of mild solutions and on their continuous dependence on the initial data. Since they are rather difficult to localize in the existing literature we have included some results on Nemitskii-operators on spaces of Höldercontinuous functions which are necessary when translating a parabolic equation into the language of abstract evolution equations. We establish a simple result on globality of solutions and give applications to the question when L_∞-a priori bounds for solutions of parabolic equations imply their globality. Finally, in Section 18, we prove some results on continuous and differentiable parameter dependence of solutions to parameter dependent semilinear evolution equations.

Chapter V deals with time-periodic semilinear evolution equations. We review, as a motivation, some results from the qualitative theory for semilinear autonomous equations. We then proceed to introduce the class of time-periodic equations we shall be interested in, define the concept of the period-map for such problems, and establish the equivalence between fixed-point of this map and periodic solutions of the original equation. After defining the basic concepts from stability theory we prove the equivalence between the Ljapunov stability of a periodic solution and its stability as a fixed-point of the period-map. We also prove the principles of linearized stability and linearized instability. We base our proof on stability results for fixed-points of mappings. We conclude this chapter with some results on when stability established in a weak norm implies stability with respect to a stronger norm.

In the last chapter we show how the abstract theory may be applied to concrete equations arising from the applied sciences. Here we consider semilinear parabolic initial-boundary value problems on bounded subdomains of $\mathbb{R}^n$ and semilinear parabolic initial value problems on the whole of $\mathbb{R}^n$. In the last section we describe how to treat a model from epidemiology, for which it is not immediately clear that it fits in the parabolic context.

We assume a working knowledge in functional analysis, calculus in Banach spaces and semigroup theory. Furthermore, we will use some of the well established theory on partial differential equations in the applied chapters. We have striven to provide precise references for the material which we only quote, whenever we feel it is not standard.

Most of our results are probably known to the specialists and though they often seem optimal for the range they cover, we have not aimed at greatest generality. To our knowledge there is no presentation of this kind of theory on this level and we hope to render the more general literature (see the references throughout the text) accessible to the reader interested in applications. It is also our intention that these notes provide the abstract setting for the theory contained in [**67**]. Most of the inspiration and information for this work were drawn from the lectures and papers by H. Amann, for the abstract chapters, and by P. Hess, for the applied ones. During all our years in Zürich we were able to profit from their vast knowledge. To both of them we express our gratitude. As is probably true for every book the fact that it finally appears in published form is the product of many circumstances and the merit not only of the authors. Various people, friends and colleagues, have made at different stages and in different ways a contribution to this work. Thanking them here is more than a mere perfunctory act. We are specially indebted to: F. Baur, J. Escher, D. Guidetti, U. Hassler, M. Hieber, A. Johnston, G. Schätti, G. Simonett and A. Wyler.

Finally, we mention that the camera ready copy of this text was typset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

0. General notation

We start by fixing the notation to which we shall adhere in this notes and recalling some basic functional analytic facts which shall be frequently used. The reader is advised to skim this section to become acquainted with our terminology and use it as a reference whenever needed. For the notation on function spaces we refer to the appendix.

A. Sets: We shall use standard set theoretic notation without further comment. If X and Y are sets we denote by Y^X the set of all mappings from X into Y. The terms map, mapping and function shall be used synonymously and randomly. A mapping is called *one-to-one* if it is injective and *onto* if it is surjective.

Let X and Y be arbitrary sets and $f : X \to Y$ a mapping. The *graph* of f is defined as

$$\operatorname{graph}(f) := \{(x, f(x)); x \in X\} \subset X \times Y.$$

Suppose now that $X_1 \subset X$ and $Y_1 \subset Y$ are given, and that $f(X_1) \subset Y_1$. Then, f induces in an obvious way a mapping from X_1 into Y_1 which, by abuse of notation, shall be denoted by the same symbol f. Thus, if $\mathcal{F}_1(X, Y)$ and $\mathcal{F}_2(X_1, Y_1)$ are classes of Y and Y_1-valued functions defined on X and X_1, respectively, we shall write

$$\mathcal{F}_1(X, Y) \cap \mathcal{F}_2(X_1, Y_1)$$

for the class of Y-valued functions f in $\mathcal{F}_1(X, Y)$ such that $[x \mapsto f(x)] \in \mathcal{F}_2(X_1, Y_1)$.

The sets of all natural, integer, rational, real and complex numbers shall be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We set $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. It is clear how to define $\mathbb{Q}^*$, $\mathbb{R}^*$ and $\mathbb{C}^*$. Sometimes the symbol $\mathbb{K}$ will be used to denote a fixed choice of either one of the fields $\mathbb{R}$ or $\mathbb{C}$. If λ is a complex number we write $\operatorname{Re}(\lambda)$ for its real part, $\operatorname{Im}(\lambda)$ for its imaginary part and $\bar{\lambda}$ for its conjugate.

If P is a property which a complex number may enjoy or not we write $[P(\lambda)]$ for the set of all complex numbers satisfying that property, e.g.

$$[\operatorname{Re}\lambda \le 0] = \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \le 0\}.$$

If n is an integer and $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are elements of $\mathbb{K}^n$ we set

$$(x|y) := \sum_{k=1}^{n} x_k \bar{y}_k$$

and

$$|x| := \sqrt{(x|x)}.$$

Thus, $(\cdot\,|\,\cdot)$ and $|\cdot|$ denote the euclidean inner product and the euclidean norm on $\mathbb{K}^n$, respectively. Finally we shall use the notation

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n;\ x_i \geq 0\ i = 1, 2, \ldots, n\}.$$

If above $n = 1$ we shall just write $\mathbb{R}_+$.

B. Topological spaces: Suppose X is a topological space and let A be a subset of X. We shall often think of A as a topological space by its own right, endowing it with the relative topology. We write

$$\overset{\circ}{A}, \quad \bar{A}, \quad \partial A$$

to mean the *interior*, the *closure* and the *boundary* of A in X, respectively. If the choice of the reference space X is not obvious from the context we shall also write $\mathrm{int}_X(A)$, $\mathrm{cl}_X(A)$ and $\partial_X(A)$ for the above sets. The subset A is said to be *relatively compact in* X if its closure $\bar{A}$ is a compact subset of X.

If Y is a further topological space we denote the set of all continuous functions from X into Y by

$$C(X, Y).$$

If X is a metric space with metric d we set for any $x_0 \in X$ and $\varepsilon > 0$

$$\mathbb{B}_X(x_0, \varepsilon) := \{x \in X;\ d(x_0, x) < \varepsilon\}.$$

The open set $\mathbb{B}_X(x_0, \varepsilon)$ is called the *open ball centered at* x_0 *with radius* ε. The *closed ball* is defined similarly substituting '$<$' by '$\leq$'. We usually write $\mathbb{B}_X$ instead of $\mathbb{B}_X(0, 1)$, and just $\mathbb{B}^n$ whenever $X = \mathbb{R}^n$, $n \geq 1$.

C. Linear operators: Let now $\mathbb{K}$ be either one of the fields $\mathbb{R}$ or $\mathbb{C}$. Suppose that X and Y are linear metric spaces (mostly they will be Banach spaces). The linear space of all continuous linear operators from X into Y shall be denoted by $\mathcal{L}(X, Y)$. We shall write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$.

A subset B of a linear metric space Z is called *topologically bounded* (or just *bounded*) if it is absorbed by any zero-neighbourhood, i.e. if to each zero-neighbourhood U there exists a scalar λ such that $B \subset \lambda U$. Observe that in general this is not equivalent to B being bounded with respect to the metric in Z, i.e. $\sup_{x\in B} d(0, x) < \infty$. But if Z is a normed space then the boundedness of B is equivalent to its *norm boundedness*, i.e. $\sup_{x\in B} \|x\| < \infty$.

Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on the vector space Z. Recall that $\|\cdot\|_1$ is said to be *weaker* than $\|\cdot\|_2$ if $\|x\|_1 \leq c\|x\|_2$ holds for some constant $c > 0$ and every $x \in Z$. In this case $\|\cdot\|_2$ is said to be *stronger* than $\|\cdot\|_1$. These norms are said to be *equivalent* if $\|\cdot\|_2$ is both stronger and weaker than $\|\cdot\|_1$. In this case they generate the same topology on Z.

A linear operator $T: X \to Y$ is called *bounded* if it maps bounded subsets of X into bounded subsets of Y. Then T is bounded if and only if T is continuous. Because of this we shall use terms bounded and continuous linear operators synonymously.

A linear operator $T \in \mathcal{L}(X,Y)$ is called *invertible* if it is bijective and its inverse $T^{-1}: Y \to X$ is bounded. If X and Y are *Fréchet spaces*, i.e. complete linear metric locally convex spaces, the open mapping theorem implies that $T \in \mathcal{L}(X,Y)$ is invertible if and only if it is bijective. We denote the set of all invertible bounded operators from X to Y by $\mathrm{Isom}(X,Y)$ and set $\mathcal{GL}(X) := \mathrm{Isom}(X,X)$. Note that $\mathrm{Isom}(X,Y)$ is not a subspace of $\mathcal{L}(X,Y)$ but only an open subset.

An operator $T \in \mathcal{L}(X,Y)$ is said to be *compact* if it maps bounded subsets of X into relatively compact subsets of Y. We denote the subspace of $\mathcal{L}(X,Y)$ consisting of all compact operators by $\mathcal{K}(X,Y)$ and put $\mathcal{K}(X) := \mathcal{K}(X,X)$. Consider a third linear metric space Z. If $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$ then $ST := S \circ T \in \mathcal{K}(X,Z)$ whenever T or S is compact. This shall be referred to as the *stability property* of compact operators.

We shall often deal with linear operators A taking values in Y which are not defined on the whole space X but only on a subspace $D(A)$ of X, called the *domain of definition* of A:

$$A: X \supset D(A) \to Y.$$

Such an operator is called *closed* if $\mathrm{graph}(A)$ is a closed subset of $X \times Y$ with respect to the product topology. Furthermore, A is said to be *densely defined* if $D(A)$ is a dense subset of X.

Suppose we can write X as the topological direct sum of two subspaces X_1 and X_2, i.e.

$$X = X_1 \oplus X_2.$$

This decomposition is said to *reduce* the linear operator $A: X \supset D(A) \to X$ if both X_1 and X_2 are invariant under A, i.e. $A(X_i \cap D(A)) \subset X_i$ for $i = 1,2$. If this is the case we can write in an obvious way

$$A = A_1 \oplus A_2\,,$$

where the operators $A_i: X_i \supset D(A_i) \to X_i$ are defined by $D(A_i) := X_i \cap D(A)$ and $A_i x := Ax$ for any $x \in X_i$, $i = 1,2$. If A is closed or bounded this property is inherited by both A_1 and A_2.

If X is a normed space we shall write $\|\cdot\|$ for its norm. This means that if we consider several Banach spaces we shall denote all norms by the same symbol $\|\cdot\|$. If there should be any ambiguity we shall provide the norms with subscripts such as $\|\cdot\|_X$. Suppose now that X and Y are normed spaces. Then we can define a norm on $\mathcal{L}(X,Y)$ by setting

$$\|T\| := \sup_{\|x\|\le 1} \|Tx\| = \inf\{c \ge 0;\ \|Tx\| \le c\|x\| \text{ for all } x \in X\}.$$

The topology on $\mathcal{L}(X,Y)$ induced by this norm is called the *uniform operator topology*. Convergence of a sequence of operators or continuity of a function taking values in $\mathcal{L}(X,Y)$ shall always be understood with respect to this topology, unless explicitly stated. If Y is a Banach space then $\mathcal{L}(X,Y)$ equipped with the above norm is also a Banach space.

Observe that if X is a Banach space, $\mathcal{L}(X)$ is actually a Banach algebra and, by the stability property of compact operators, $\mathcal{K}(X)$ a two-sided ideal in this algebra. The unit element of $\mathcal{L}(X)$, i.e. the identity map, shall be denoted by $\mathbb{1}_X$ or just $\mathbb{1}$ if no confusion seems likely.

Suppose now that $A\colon X \supset D(A) \to Y$ is a linear operator. Then we can define a norm on $D(A)$ by setting

$$\|x\|_{D(A)} := \|x\| + \|Ax\|,$$

for all $x \in D(A)$. We shall always think of $D(A)$ as being equipped with this norm which is called the *graph-norm* on $D(A)$. The closedness of A is then equivalent to $D(A)$ being a Banach space. If A is bijective from its domain of definition to Y and its inverse A^{-1} is a bounded operator from Y into X then

$$|\!|\!|x|\!|\!|_{D(A)} := \|Ax\| \qquad \text{for all } x \in D(A)$$

defines a norm which is equivalent to the graph-norm on $D(A)$.

Finally, for any linear operator $A\colon X \supset D(A) \to Y$ we set

$$\ker(A) := \{x \in D(A);\ Ax = 0\} \qquad \text{and} \qquad \operatorname{im}(A) := \{Ax;\ x \in D(A)\}.$$

The subspaces $\ker(A)$ of X and $\operatorname{im}(A)$ of Y are called *kernel* (or *null-space*) and *range* (or *image*) of A, respectively.

D. Imbeddings: Suppose that X and Y are linear metric spaces. If $X \subset Y$ holds (as sets) and the inclusion map $i\colon X \to Y$ is continuous we say that X is *continuously imbedded* in Y and write

$$X \hookrightarrow Y.$$

This is equivalent to saying that if X is endowed with the relative topology of Y this topology is weaker than the original one.

X is said to be *densely imbedded* in Y if $X \hookrightarrow Y$ and X is a dense subset of Y. In this case we write

$$X \overset{d}{\hookrightarrow} Y.$$

If $X \hookrightarrow Y$ holds and the inclusion map i is compact then X is said to be *compactly imbedded* in Y. We write for a compact imbedding and a compact dense imbedding

$$Y \overset{c}{\hookrightarrow} X \qquad \text{and} \qquad Y \overset{d}{\overset{c}{\hookrightarrow}} X,$$

respectively.

Whenever X is continuously imbedded in Y and Z is another linear metric space we can view $\mathcal{L}(Y, Z)$ as a linear subspace of $\mathcal{L}(X, Z)$ by identifying $T \in \mathcal{L}(Y, Z)$ with $T \circ i \in \mathcal{L}(X, Z)$. If X is densely imbedded this correspondence is one-to-one. If X is compactly imbedded in Y then $\mathcal{L}(Y, Z) \subset \mathcal{K}(X, Z)$ by the stability property of compact operators.

Suppose that X and Y are normed spaces and that $X \hookrightarrow Y$ holds. Then we have

$$\|x\|_Y \leq \|i\|_{\mathcal{L}(X,Y)} \|x\|_X$$

for all $x \in X$. We call $\|i\|_{\mathcal{L}(X,Y)}$ the *imbedding constant* of $X \hookrightarrow Y$.

Finally if two normed spaces X and Y are equal as sets and their norms are equivalent we write

$$X \doteq Y.$$

Let X and Y be Fréchet spaces satisfying $X \hookrightarrow Y$ and let $A\colon Y \supset D(A) \to Y$ be a closed linear operator. The operator $A_X : X \supset D(A_X) \to X$, called the X-*realization* of A, is defined by

$$D(A_X) = \{x \in D(A) \cap X;\ Ax \in X\} \qquad \text{and} \qquad A_X x = Ax \text{ for all } x \in D(A_X).$$

Observe that A_X is a closed operator. The same notions are used if Y is not a Fréchet space but only a locally convex space (for example the space $\mathcal{D}'$ of Distributions when dealing with differential operators in function spaces).

E. Duality: Let X be a Banach space. By a *linear functional* on X we mean a linear operator from X into $\mathbb{K}$. The *topological dual space* of X is the Banach space $X' := \mathcal{L}(X, \mathbb{K})$. If $x' \in X'$ and $x \in X$ we write

$$<x', x> := x'(x).$$

Let Y be a further Banach space and $T \in \mathcal{L}(X, Y)$. The *adjoint* or *dual* operator of T is the uniquely determined operator $T' \in \mathcal{L}(Y', X')$ satisfying

$$<T'y', x> = <y', Tx>$$

for all $y' \in Y'$ and $x \in X$.

If M is a subspace of X and N a subspace of X' we define their *annihilators* to be the closed subspaces

$$M^{\perp} := \{x' \in X';\ <x', x> = 0 \text{ for all } x \in M\}$$

$$^{\perp}N := \{x \in X;\ <x', x> = 0 \text{ for all } x' \in N\}$$

of X' and X, respectively. The following identities hold for any operator $T \in \mathcal{L}(X, Y)$ (cf. [**106**], Theorem 4.12)

$$\ker(T') = \operatorname{im}(T)^{\perp} \qquad \text{and} \qquad \ker(T) = {}^{\perp}\operatorname{im}(T').$$

F. Spectral theory: Let X be a Banach space and $A\colon X \supset D(A) \to X$ a densely defined closed linear operator. If the space is real the notions below are defined in the context of its complexification $X_{\mathbb{C}}$. This Banach space consists of the formal expressions of the form $z := x + iy$ with $x, y \in X$. The operations on $X_{\mathbb{C}}$ are defined as

$$z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$$

and

$$\lambda z := (\lambda_1 x - \lambda_2 y) + i(\lambda_2 x + \lambda_1 y)$$

whenever $z, z_1, z_2 \in X_{\mathbb{C}}$ and $\lambda + \lambda_1 + i\lambda_2 \in \mathbb{C}$. The norm on $X_{\mathbb{C}}$ is given by

$$\|z\| := \max_{0 \le \phi \le 2\pi} \|\cos(\phi)x + \sin(\phi)y\|.$$

As was to be expected $\mathbb{R}_{\mathbb{C}} = \mathbb{C}$. The complexification of A is then the operator $A_{\mathbb{C}}\colon X_{\mathbb{C}} \supset D(A_{\mathbb{C}}) \to X_{\mathbb{C}}$ defined by $D(A_{\mathbb{C}}) := \{\ z = x + iy;\ x, y \in D(A)\ \}$ and $A_{\mathbb{C}} z := Ax + iAy$ for $z \in D(A_{\mathbb{C}})$.

The *resolvent set*, $\varrho(A)$, of A is the open subset of $\mathbb{C}$ defined by

$$\varrho(A) := \{\lambda \in \mathbb{C};\ (\lambda - A)^{-1} \text{ exists and lies in } \mathcal{L}(X)\}.$$

Here we use the notation $\lambda - A := \lambda \mathbb{1} - A$. The *spectrum*, $\sigma(A)$, of A is the complement of the resolvent set in $\mathbb{C}$, i.e. $\sigma(A) := \mathbb{C} \setminus \varrho(A)$. Hence, $\sigma(A)$ is a closed set. The spectrum of A can be writen as the disjoint union of the following sets

$$\sigma_p(A) := \{\lambda \in \mathbb{C};\ (\lambda - A) \text{ is not injective}\},$$

$$\sigma_c(A) := \{\lambda \in \mathbb{C};\ (\lambda - A) \text{ is injective, } \operatorname{im}(\lambda - A) \text{ dense in } X,\ (\lambda - A)^{-1} \text{ not bounded}\},$$

$$\sigma_r(A) := \{\lambda \in \mathbb{C};\ (\lambda - A) \text{ is injective and } \operatorname{im}(\lambda - A) \text{ is not dense in } X\}.$$

These sets are called *point spectrum, continuous spectrum* and *residual spectrum*, respectively. An element λ of $\sigma_p(A)$ is an *eigenvalue* of A, $\ker(\lambda - A)$ is the *eigenspace* to λ and any non-zero element in it is called an *eigenvector* to λ.

If $A \in \mathcal{L}(X)$ then $\sigma(A)$ can be shown to be compact and we set

$$\mathrm{r}(A) := \sup\{|\lambda|; \lambda \in \sigma(A)\}.$$

This number is called the *spectral radius* of A. It can be calculated by the well known formula

$$\mathrm{r}(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

A subset σ_1 of $\sigma(A)$ is called *spectral set* of A if it is open and closed in $\sigma(A)$ (for instance a set consisting of an isolated point of $\sigma(A)$). If σ_1 is a compact spectral set we put $\sigma_2 := \sigma(A) \setminus \sigma_1$ and take a bounded domain $U \in \mathbb{C}$, such that

$$\sigma_1 \in U, \quad \sigma_2 \in \mathbb{C} \setminus \bar{U},$$

and

$$\partial U = \bigcup_{j=i}^{m} \Gamma_j\,,$$

where $(\Gamma_j)_{1\leq m}$ is a finite collection of disjoint smooth Jordan-curves which are positively oriented with respect to U. Then we can define the following *Dunford-type* integral:

$$P_1 := \frac{1}{2\pi i}\int_{\partial U} (\lambda - A)^{-1}\, d\lambda \qquad \text{in } \mathcal{L}(X).$$

The operator P_1 is a projection which is independent of the particular choice of a domain U with the above properties. We call P_1 the *spectral projection of A with respect to σ_1*. Setting $X_1 := P_1(X)$ and $X_2 := (\mathbb{1} - P_1)(X)$ we obtain a decomposition

$$X = X_1 \oplus X_2\,,$$

which reduces the operator A

$$A = A_1 \oplus A_2\,.$$

Furthermore, we have

$$\sigma(A_i) = \sigma_i,\ i = 1, 2.$$

G. Derivatives: Let X and Y be Banach spaces. If the function $f\colon U \to Y$, defined on the open subset U of X, is (Fréchet)-differentiable at a point $x_0 \in U$ then its derivative at x_0 is a linear operator in $\mathcal{L}(X, Y)$ and we write either $Df(x_0)$ or $f'(x_0)$ to denote it. Suppose $X := X_1 \times X_2$ and $U := U_1 \times U_2$, where U_i is an open subset of the Banach space X_i, $i = 1, 2$. If $f\colon U \to Y$ is differentiable at $(x_0^1, x_0^2) \in U$ we denote its i-th partial derivative, which is an operator in $\mathcal{L}(X_i, Y)$ by $D_i f(x_0^1, x_0^2)$, $i = 1, 2$.

If $f\colon \Omega \to \mathbb{R}$ is a differentiable function defined on an open subset Ω of $\mathbb{R}^n$, $n \geq 1$, we write for any $x_0 \in \Omega$ and $i = 1, \dots, n$

$$\partial_i f(x_0) := D_i f(x_0).$$

The *gradient* of f at $x_0 \in \Omega$ shall be denoted by

$$\operatorname{grad} f(x_0) := \nabla f(x_0) := \big(\partial_1 f(x_0), \dots, \partial_n f(x_0)\big) \in \mathbb{R}^n.$$

The *Laplace operator* or *Laplacian* of f at x_0 is defined as

$$\Delta f(x_0) := \partial_1^2 f(x_0) + \cdots + \partial_n^2 f(x_0) \in \mathbb{R}.$$

We call an element $\alpha = (\alpha_1, \dots, \alpha_n)$ of $\mathbb{N}^n$ a *multiindex* (*of rank* n). The *order* of α is given by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $\alpha \in \mathbb{N}^n$. If $f\colon \Omega \to \mathbb{R}$ is sufficiently smooth in $x_0 \in \Omega$, we write

$$\partial^\alpha f(x_0) = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f(x_0).$$

If α and β are two multiindices we write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, \dots, n$. We define

$$\alpha! := \prod_{i=1}^n \alpha_i! \qquad \text{and} \qquad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}\ ,$$

where $\beta \leq \alpha$ and $(\alpha - \beta)_i = \alpha_i - \beta_i$ for all $i = 1, \dots, n$. Let now $f, g\colon \Omega \to \mathbb{R}$ be two functions. Then, *Leibniz's rule*

$$\partial^\alpha \big(fg\big)(x_0) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x_0) \partial^{\alpha - \beta} g(x_0)$$

holds, whenever it makes sense.

If $f\colon \Omega \times I \to \mathbb{R}$ is defined on the product of an open subset Ω of $\mathbb{R}^n$ and an open interval I in $\mathbb{R}$, we think of the point $(x, t) \in \Omega \times I$ as consisting of the *space-variable* x and the *time-variable* t. Furthermore we write

$$\partial_t f(x_0, t_0) := D_{n+1} f(x_0, t_0),$$

$$\partial_i f(x_0, t_0) := D_i f(x_0, t_0),$$

$$\operatorname{grad} f(x_0, t_0) := \nabla f(x_0, t_0) := \big(\partial_1 f(x_0, t_0), \dots, \partial_n f(x_0, t_0)\big)$$

and

$$\Delta f(x_0, t_0) := \partial_1^2 f(x_0, t_0) + \cdots + \partial_n^2 f(x_0, t_0)$$