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Nonlinear Continuum Mechanics of Solids

Fundamental mathematical and physical concepts

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Fundamental Mathematical and Physical Concepts

With 35 figures and 5 tables

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Preface

After the fast development of computational methods in the last decades, attention in today's research in continuum and computational mechanics is focusing increasingly on the refinement of theoretical formulations. Nowadays, the problem ist not so much to transform existing theoretical models into numerical codes, but to improve the accuracy of these models in various fields (material modelling, large strain analysis, damage mechanics, optimization of nonlinear structures, etc.) in order to ensure a more realistic simulation especially of nonlinear phenomena. Evidently, research on such complex topics requires a profound understanding of nonlinear continuum mechanics and expertise in tensor analysis.

The content of this book reflects essentially the lectures given by the authors at Ruhr-University Bochum, Technical University Aachen and Technical University Lille for graduate students of engineering and material sciences, applied mathematicians and research engineers wishing to be brought rapidly within reach of their specific research area. Hence, fundamental concepts which are of general interest and not special topics have been found to be relevant for the presentation. The authors are aware of the gap between *"what is taught in classical engineering education and what is required by research on current topics of nonlinear structural and continuum mechanics"*. This book attempts to bridge this gap.

In this book equations are developed in absolute tensor notation which is used almost exclusively in modern literature of continuum mechanics. This notation provides a general and elegant formulation of the theoretical background particularly for nonlinear problems and is, moreover, of considerable help in transforming the theory into numerical codes. In contrast to many other works on tensor calculus, here the aim is to present the fundamentals of continuum mechanics of solids together with the mathematical background in an unified description. Accordingly, the mathematical tools are presented so as to enable the reader to study the book without permanent reference to other works.

The first chapter presents the basic rules of tensor calculus in absolute notation and introduces the special tensors relevant for continuum mechanics. It also deals extensively with the eigenvalue problems of second-order tensors, the orthogonal and rotation tensors and the differentiation rules of tensors. Thus it involves all basic mathematical concepts needed in the sequel.

The second chapter is devoted to a detailed description of deformations of solids under systematical consideration of geometrical nonlinearities. Here, various deformation and strain measures are defined, their mechanical interpretation is given through the corresponding eigenvalue problems and a systematical classification of the strain tensors is presented. This section involves also further relevant topics such as pull-back and push-forward operations, and the definition of the rate of deformation tensor as well as isotropic tensor functions the last being of special significance for material modelling. The detailed derivations in this part should enable the reader to get experienced with the tensor calculus. An in-depth study of this chapter together with the foregoing one is recommended for an easy understanding of the book. Formulae and definitions of tensor algebra in index notation, which are prerequisites for chapter 1, are summarized in appendix 1.

Chapter 3 starts with the definition of the CAUCHY stress tensor where emphasis is placed on its mechanical interpretation. Subsequently, various stress tensors are defined by purely mathematical transformations and then shown to be energy conjugate to the strain tensors from the previous chapter through the rate of internal energy. A precise definition of the internal energy is, however, given in chapter 5 in connection with the law of conservation of energy.

The notion of the material time derivative is explained in chapter 4 and then applied to define the velocity and the acceleration vector. The material time derivatives of some geometrical variables such as volume, surface and line elements are also given in this chapter.

Chapter 5 presents in a systematic way the balance laws: conservation of mass, balance of momentum, balance of moment of momentum, balance of kinetic energy and conservation of energy. Equations of motion are obtained as local formulation of balance of momentum. Similarly, the symmetry of the CAUCHY stress tensor introduced in chapter 3 as a postulate is proved through the local formulation of balance of moment of momentum. This chapter closes with the derivation of the principle of virtual work as weak formulation of the equations of motion and the dynamic boundary conditions.

Material modelling at large elastic strains is extensively discussed in chapter 6. The discussion starts with the general principles to be considered in formulating material laws and the definition of objective tensors. Hyperelastic materials are defined first in a general form and then particular attention is paid to isotropic materials. In this context many practically important material models are presented. Finally, some useful connections between them are established through linearization.

Each chapter includes a number of applications in order to help the reader to get experienced with the theory. Some of them present also important results needed in the subsequent derivations.

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August 1999 Y. Basar and D. Weichert

Contents

1 Mathematical fundamentals

This section defines tensors as invariant quantities and introduces tensorial operations in absolute notation. Emphasis is given to the definition of some special tensors playing an important role in continuum mechanics. In addition, some useful results such as the definition of the gradient and the divergence of tensors are presented. This section aims to present the mathematical background for an easy understanding of the following sections.

1.1 Simple tensors

First-order tensors. In this section we deal with simple tensors whose definition is based on physical or geometrical (invariant) vectors of the 3D-Euclidean space E3. We use curvilinear coordinates Θ^i to define points of E3, $\mathbf{g}_i$ and $\mathbf{g}^i$ denoting the associated covariant and contravariant base vectors. Vectors will in general be denoted by bold lower-case letters, e.g. by **a**, **b**, ... Their components with respect to a new coordinate system $\bar{\Theta}^i$ will be presented by $(\bar{...})$ and those referring to the initial coordinate system Θ^i without any mark, thus

$$\mathbf{a} = a_i \mathbf{g}^i = a^i \mathbf{g}_i = \bar{a}_i \bar{\mathbf{g}}^i = \bar{a}^i \bar{\mathbf{g}}_i \quad . \tag{1.1.1}$$

The variables a_i, a^i are first-order tensor components, while the vector **a** itself being independent of any coordinate system is said to form a *first-order tensor*. Before extending this concept to the definition of higher-order tensors we first recall the usual vectorial operations with their appropriate notations:

scalar product: $$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = (a_i \mathbf{g}^i) \cdot (b^j \mathbf{g}_j) = a_i b^i = a^i b_i \ , \tag{1.1.2}$$

vectorial product: $$\mathbf{c} = \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = \varepsilon_{ijk} a^i b^j \mathbf{g}^k = \varepsilon^{ijk} a_i b_j \mathbf{g}_k \ , \tag{1.1.3}$$

mixed product: $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \varepsilon_{ijk} a^i b^j c^k = \varepsilon^{ijk} a_i b_j c_k \ . \tag{1.1.4}$$

We see that the first and the last operation lead to invariant scalar-valued quantities, *zero-order tensors*, while the result of the vectorial product is a new vector **c** in E3.

Tensorial product. We introduce a new operation for arbitrary vectors, the so-called *tensorial (dyadic) product.* If this operation (notation with $\otimes$) is applied to invariant vectors **a** and **b** the result will be a new invariant quantity **S** called a *second-order simple tensor* or *dyad.* Thus

$$\mathbf{S} = \mathbf{a} \otimes \mathbf{b} \ . \tag{1.1.5}$$

Similarly, simple tensors of arbitrary order *(polyads)* can be constructed, e.g. a third-order tensor:

$$\mathbf{R} = \mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e} \ . \tag{1.1.6}$$

By definition the tensorial product is *not commutative*

$$\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a} \ , \tag{1.1.7}$$

but it is supposed to satisfy the following requirements:

$$\textit{the distributive rule:} \quad \mathbf{a} \otimes (\mathbf{b} + \mathbf{c}) = \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{c} \ , \tag{1.1.8}$$

$$\textit{the associative rule:} \quad (\alpha\,\mathbf{a}) \otimes \mathbf{b} = \mathbf{a} \otimes (\alpha\,\mathbf{b}) = \alpha\,(\mathbf{a} \otimes \mathbf{b}) \ , \tag{1.1.9}$$

α being a scalar.

In terms of vector components defined according to (1.1.1) we obtain from (1.1.5) with the bases $\mathbf{g}_i$ and $\mathbf{g}^i$

$$\begin{aligned} \mathbf{S} &= a^i b^j \mathbf{g}_i \otimes \mathbf{g}_j = a^i b_j \mathbf{g}_i \otimes \mathbf{g}^j = a_i b^j \mathbf{g}^i \otimes \mathbf{g}_j = a_i b_j \mathbf{g}^i \otimes \mathbf{g}^j \\ &= S^{ij} \mathbf{g}_i \otimes \mathbf{g}_j = S^{i}_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j = S_{i}^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j = S_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \end{aligned} \tag{1.1.10}$$

indicating that a second order tensor can be equivalently represented by four different sets of components S^{ij}, $S^{i}_{\cdot j}$, $S_{i}^{\cdot j}$, S_{ij}.

Simple contraction. Tensors of arbitrary orders, e.g. **S** (1.1.5) and **R** (1.1.6), may be related by the *simple contraction*, denoted as **S R** (without any mark). This operation provides the scalar multiplication of adjacent vectors of contributing tensors such that

$$\mathbf{S}\,\mathbf{R} = (\mathbf{a} \otimes \mathbf{b})\,(\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) = (\mathbf{b} \cdot \mathbf{c})\,(\mathbf{a} \otimes \mathbf{d} \otimes \mathbf{e}) \ . \tag{1.1.11}$$

Thus, it leads to a tensor **S R** the order of which is twice less than the sum of the orders of the participants **S** and **R**. The simple contraction obeys by virtue of (1.1.8), (1.1.9) and (1.1.11) the following rules

$$\textit{the distributive rule:} \quad \mathbf{R}\,(\mathbf{S} + \mathbf{T}) = \mathbf{R}\,\mathbf{S} + \mathbf{R}\,\mathbf{T} \ , \tag{1.1.12}$$

$$(\mathbf{S} + \mathbf{T})\,\mathbf{R} = \mathbf{S}\,\mathbf{R} + \mathbf{T}\,\mathbf{R} \ , \tag{1.1.13}$$

$$\textit{the associative rule:} \quad (\mathbf{S}\,\mathbf{T})\,\mathbf{R} = \mathbf{S}\,(\mathbf{T}\,\mathbf{R}) \ , \tag{1.1.14}$$

$$\alpha\,(\mathbf{S}\,\mathbf{T}) = (\alpha\,\mathbf{S})\,\mathbf{T} = \mathbf{S}\,(\alpha\,\mathbf{T}) \ , \tag{1.1.15}$$

where α is a scalar. Evidently, in equations (1.1.12) and (1.1.13) the tensors **S** and **T** are supposed to be of the same order. In general the simple contraction is *not commutative*:

$$\begin{aligned} \mathbf{R}\,\mathbf{S} &= (\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e})\,(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{e} \cdot \mathbf{a})\,(\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{b}) \\ &\neq \mathbf{S}\,\mathbf{R} = (\mathbf{a} \otimes \mathbf{b})\,(\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) = (\mathbf{b} \cdot \mathbf{c})\,(\mathbf{a} \otimes \mathbf{d} \otimes \mathbf{e}) \ . \end{aligned}$$

According to the above mentioned rules the following relations hold:

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{u} = (\mathbf{b} \cdot \mathbf{u})\,\mathbf{a} = (\mathbf{u} \cdot \mathbf{b})\,\mathbf{a} \ ,$$

$$(\mathbf{a} \otimes \mathbf{b})\,(\mathbf{u} + \mathbf{v}) = [\mathbf{b} \cdot (\mathbf{u} + \mathbf{v})]\,\mathbf{a} = (\mathbf{b} \cdot \mathbf{u})\,\mathbf{a} + (\mathbf{b} \cdot \mathbf{v})\,\mathbf{a} \ . \tag{1.1.16}$$

If **S** and **R** are first-order tensors, the simple contraction (1.1.11) corresponds to the scalar-product of vectors and is only in this case commutative.

Double contraction. A further important operation applicable to higher-order tensors **S** and **R** is the *double contraction*, denoted by **S**:**R**. In this case two scalar products are to be carried out in the form

$$\mathbf{S} : \mathbf{R} = (\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d})\,\mathbf{e} \tag{1.1.17}$$

leading to a tensor **S**:**R** whose order is four times less than the sum of the orders of the participants **S** and **R**.

The double contraction satisfies the following rules:

$$\textit{the distributive rule:}\ \ \mathbf{R} : (\mathbf{S} + \mathbf{T}) = \mathbf{R} : \mathbf{S} + \mathbf{R} : \mathbf{T} \ , \tag{1.1.18}$$

$$\textit{the associative rule:}\ \ (\alpha\,\mathbf{R}) : \mathbf{S} = \mathbf{R} : (\alpha\,\mathbf{S}) = \alpha\,(\mathbf{R} : \mathbf{S}) \ . \tag{1.1.19}$$

Generally the double contraction is *not commutative*:

$$\mathbf{R} : \mathbf{S} = (\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{e}) : (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{e} \cdot \mathbf{b})\,(\mathbf{d} \cdot \mathbf{a})\,\mathbf{c} \neq \mathbf{S} : \mathbf{R} = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d})\,\mathbf{e} \ .$$

An exception is however the case where the participant tensors **S** and **R** are both of second-order:

$$\mathbf{S} : \mathbf{R} = (\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d})$$

$$= (\mathbf{c} \cdot \mathbf{a})\,(\mathbf{d} \cdot \mathbf{b}) = (\mathbf{c} \otimes \mathbf{d}) : (\mathbf{a} \otimes \mathbf{b}) = \mathbf{R} : \mathbf{S} \ . \tag{1.1.20}$$

Further we may write for an arbitrary second-order tensor **S**

$$\mathbf{S} : (\mathbf{c} \otimes \mathbf{d}) = \mathbf{c} \cdot (\mathbf{S}\,\mathbf{d}) = (\mathbf{c}\,\mathbf{S}) \cdot \mathbf{d} = \mathbf{c}\,\mathbf{S}\,\mathbf{d} \ , \tag{1.1.21}$$

which can easily be proved by using for **S** the expression (1.1.5). Thus,

$$\mathbf{S} : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) \ , \tag{1.1.22}$$

$$\mathbf{c} \cdot (\mathbf{S}\,\mathbf{d}) = \mathbf{c} \cdot [(\mathbf{a} \otimes \mathbf{b})\,\mathbf{d}] = [(\mathbf{b} \cdot \mathbf{d})\,\mathbf{a}] \cdot \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) \ . \tag{1.1.23}$$

1.2 General tensors

Tensors as invariant quantities. We now consider arbitrary tensor components

$$a_i \ , \ S_{ij} \ , \ T_{ijk}$$

defined with respect to a curvilinear coordinate system Θ^i. The values of the above components in connection with new coordinates $\bar{\Theta}^i$ will be presented by $(\bar{..})$. We recall that both kinds of components S_{ij} and $\bar{S}_{ij}$ are related by

$$\bar{S}_{ij} = \frac{\partial \Theta^k}{\partial \bar{\Theta}^i} \frac{\partial \Theta^l}{\partial \bar{\Theta}^j} S_{kl} \,, \quad S_{ij} = \frac{\partial \bar{\Theta}^k}{\partial \Theta^i} \frac{\partial \bar{\Theta}^l}{\partial \Theta^j} \bar{S}_{kl} \,. \quad \text{(summation over k and l)} \qquad (1.2.1)$$

Here and in the sequel it is assumed that there exists a sufficiently smooth, one to one mapping between the system of coordinates Θ^i and $\bar{\Theta}^i$.

In (1.1.10) we have already observed that second-order tensor components $S^{ij} = a^i b^j$ form by using the basis $\mathbf{g}_i \otimes \mathbf{g}_j$ an invariant quantity $\mathbf{S}$ called tensor. Using suitable bases

$$\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes \ldots \qquad (1.2.2)$$

as tensorial product of the base vectors $\mathbf{g}_i$, $\mathbf{g}_j$, ... the above idea can readily be extended to associate arbitrary sets of tensor components with an invariant quantity called tensor. Here we recall that expressions of the form (1.2.2) obey both of the rules (1.1.8) and (1.1.9) valid for tensorial products ($\otimes$). The property (1.1.9) has already been considered in the derivation of (1.1.10).

We now refer to S_{ij} to construct – using the basis $\mathbf{g}^i \otimes \mathbf{g}^j$ – the *second-order tensor*

$$\mathbf{S} = S_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = S_{i}^{\cdot j}\, \mathbf{g}^i \otimes \mathbf{g}_j = S_{\cdot j}^{i}\, \mathbf{g}_i \otimes \mathbf{g}^j = S^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j \,, \qquad (1.2.3)$$

where the summation rule over repeated indices applies. The relation (1.2.3) demonstrates $\mathbf{S}$ to be a general representation of four possible sets of components S_{ij}, $S_{i}^{\cdot j}$, ... Since for any coordinate transformation $\Theta^i \to \bar{\Theta}^i$ the equality

$$S_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = \bar{S}_{ij}\, \bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j = \mathbf{S} \,, \qquad (1.2.4)$$

holds we furthermore see that $\mathbf{S}$ is independent of any special coordinate system, therefore an *invariant* quantity. Accordingly, relations to be established in terms of such invariant variables will hold for arbitrary coordinate systems: a significant advantage of the *symbolic notation*. As a further example we form with T_{ijk} a *third-order tensor*

$$\mathbf{T} = T_{ijk}\, \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k = T_{\cdot jk}^{i}\, \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^k = \ldots \,. \qquad (1.2.5)$$

Our next goal is the generalisation of the operations introduced in the previous section for *polyads (simple tensors)* to the general tensors of the form (1.2.3) and (1.2.5).

Simple contraction. The application of the rule (1.1.11) to the tensors $\mathbf{S}$ and $\mathbf{R}$

$$\begin{aligned} \mathbf{S}\,\mathbf{R} &= (S_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j)\,(R_{rst}\, \mathbf{g}^r \otimes \mathbf{g}^s \otimes \mathbf{g}^t) \\ &= S_{ij}\, (\mathbf{g}^j \cdot \mathbf{g}^r)\; R_{rst}\, \mathbf{g}^i \otimes \mathbf{g}^s \otimes \mathbf{g}^t \end{aligned}$$

$$= S_{ij}\, g^{jr}\, R_{rst}\, \mathbf{g}^i \otimes \mathbf{g}^s \otimes \mathbf{g}^t$$

$$= S_{i\cdot}^{\cdot r}\, R_{rst}\, \mathbf{g}^i \otimes \mathbf{g}^s \otimes \mathbf{g}^t = S_{ir}\, R^{r}_{\cdot st}\, \mathbf{g}^i \otimes \mathbf{g}^s \otimes \mathbf{g}^t = \ldots \tag{1.2.6}$$

produces a *contraction* with respect to the last index of the first component and the first index of the second component. Thus it leads to a tensor **S R** the order of which is twice less than the order of the tensorial product **S** ⊗ **R**.

It is possible to contract any higher-order tensor from the left and right side by means of simple contractions, e.g. in the form

$$\mathbf{T\,S\,u} = (T_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j)\,(S^{st}\, \mathbf{g}_s \otimes \mathbf{g}_t)\,(u_m\, \mathbf{g}^m) = (\mathbf{g}^j \cdot \mathbf{g}_s)\,(\mathbf{g}_t \cdot \mathbf{g}^m)\, T_{ij}\, S^{st}\, u_m\, \mathbf{g}^i$$

$$= \delta^j_s\, \delta^m_t\, T_{ij}\, S^{st}\, u_m\, \mathbf{g}^i = T_{ij}\, S^{jm}\, u_m\, \mathbf{g}^i \,. \tag{1.2.7}$$

Evidently, a vector **u** can not be contracted from both sides. So, an expression of the form **T u S** is nonsense.

According to the above definitions, the validity of the following rules for simple contractions can easily be proved:

the associative rule: $(\mathbf{T\,S})\,\mathbf{R} = \mathbf{T}\,(\mathbf{S\,R})$, (1.2.8)

the distributive rule: $\mathbf{T}\,(\mathbf{R} + \mathbf{S}) = \mathbf{T\,R} + \mathbf{T\,S}$, (1.2.9)

$(\mathbf{R} + \mathbf{S})\,\mathbf{T} = \mathbf{R\,T} + \mathbf{S\,T}$, (1.2.10)

where in the last two relations **R** and **S** are supposed to be of the same order. Generally the simple contraction is *not commutative*.

Powers of second-order tensors. The simple contraction allows to define powers of a second-order tensor **S** in the form:

$$\mathbf{S}^0 = \mathbf{I}\,, \quad \mathbf{S}^1 = \mathbf{S}\,, \quad \mathbf{S}^2 = \mathbf{S\,S}\,, \quad \ldots \tag{1.2.11}$$

where by definition $\mathbf{S}^0$ is identical with the so-called identity tensor $\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i$. In accordance with (1.2.11) we also note that

$$\mathbf{S}^m\, \mathbf{S}^n = \mathbf{S}^{m+n} = \mathbf{S}^n\, \mathbf{S}^m\,, \quad (\alpha\, \mathbf{S})^m = \alpha^m\, \mathbf{S}^m\,, \quad (\mathbf{S}^m)^n = \mathbf{S}^{mn}\,. \tag{1.2.12}$$

Evaluation of tensor components. In (1.2.3) the tensor **S** is expressed in terms of its components S_{ij}. Conversely, it is possible to express S_{ij} in terms of **S** and the associated base vectors. By using the rule (1.2.6) we find

$$\mathbf{g}_i\, \mathbf{S}\, \mathbf{g}_j = \mathbf{g}_i\, (S_{mn}\, \mathbf{g}^m \otimes \mathbf{g}^n)\, \mathbf{g}_j = (\mathbf{g}_i \cdot \mathbf{g}^m)\,(\mathbf{g}^n \cdot \mathbf{g}_j)\, S_{mn} = \delta^m_i\, \delta^n_j\, S_{mn}$$

leading as final result to:

$$S_{ij} = \mathbf{g}_i\, \mathbf{S}\, \mathbf{g}_j\,, \quad S^{i}_{\cdot j} = \mathbf{g}^i\, \mathbf{S}\, \mathbf{g}_j\,, \quad S_{i\cdot}^{\cdot j} = \mathbf{g}_i\, \mathbf{S}\, \mathbf{g}^j\,, \quad S^{ij} = \mathbf{g}^i\, \mathbf{S}\, \mathbf{g}^j\,. \tag{1.2.13}$$

The above rule will often serve to evaluate tensor components if the tensor itself is given as an invariant quantity.

Application. The expression **S u** with a second-order tensor **S** and a vector **u** represents a vector. Thus its scalar product with the vector **v** can be formed as follows:

$$(\mathbf{S}\,\mathbf{u}) \cdot \mathbf{v} = \mathbf{v} \cdot (\mathbf{S}\,\mathbf{u}) = [(S^{ij}\,\mathbf{g}_i \otimes \mathbf{g}_j)\,(u_k\,\mathbf{g}^k)] \cdot v_n\,\mathbf{g}^n = S^{ij}\,u_j\,v_i \ .$$

The same result can also be obtained from the expression **v S u** so that

$$\mathbf{v}\,\mathbf{S}\,\mathbf{u} = (\mathbf{S}\,\mathbf{u}) \cdot \mathbf{v} = \mathbf{v} \cdot (\mathbf{S}\,\mathbf{u}) \ . \tag{1.2.14}$$

Application. Possible component representations of the expression **A B C v** with second-order tensors **A, B, C** and the vector **v** are:

$$A_{ij}\,B^{j}_{\cdot k}\,C^{k}_{\cdot m}\,v^m = A_{i}{}^{j}_{\cdot}\,B_{jk}\,C^{k}_{\cdot m}\,v^m = A_{i}{}^{j}_{\cdot}\,B_{j}{}^{k}_{\cdot}\,C_{km}\,v^m = \ldots \ .$$

Double contraction. The corresponding rule has already been introduced in (1.1.17) for simple tensors. Its application to arbitrary tensors **T** and **R**

$$\begin{aligned}
\mathbf{T} : \mathbf{R} &= (T^{lmn}\,\mathbf{g}_l \otimes \mathbf{g}_m \otimes \mathbf{g}_n) : (R^{ijk}\,\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k) \\
&= T^{lmn}\,(\mathbf{g}_m \cdot \mathbf{g}_i)\,(\mathbf{g}_n \cdot \mathbf{g}_j)\,R^{ijk}\,(\mathbf{g}_l \otimes \mathbf{g}_k) \\
&= T^{lmn}\,g_{mi}\,g_{nj}\,R^{ijk}\,\mathbf{g}_l \otimes \mathbf{g}_k
\end{aligned} \tag{1.2.15}$$

delivers

$$\mathbf{T} : \mathbf{R} = T^{l}_{\cdot ij}\,R^{ijk}\,\mathbf{g}_l \otimes \mathbf{g}_k = T^{lij}\,R_{ij}{}^{k}_{\cdot}\,\mathbf{g}_l \otimes \mathbf{g}_k = \ldots \ . \tag{1.2.16}$$

This operation is called *double contraction* since it reduces the order of the tensorial product **T** ⊗ **R** by four due to the two scalar products to be performed.

Generally the double contraction is *not commutative*. But if the participant tensors **A** and **B** are both of second order this property holds:

$$\mathbf{A} : \mathbf{B} = A_{ij}\,B^{ij} = B^{ij}\,A_{ij} = \mathbf{B} : \mathbf{A} \ . \tag{1.2.17}$$

From (1.2.16) the following rules can easily be deduced for the double contraction

the distributive rule: $\quad \mathbf{R} : (\mathbf{S} + \mathbf{T}) = \mathbf{R} : \mathbf{S} + \mathbf{R} : \mathbf{T} \ ,$ (1.2.18)

the associative rule: $\quad (\alpha\,\mathbf{T}) : \mathbf{S} = \mathbf{T} : (\alpha\,\mathbf{S}) = \alpha\,(\mathbf{T} : \mathbf{S}) \ ,$ (1.2.19)

where in the distributive rule **S** and **T** are supposed to be tensors of the same order. In component form relation (1.2.18) reads as

$$R^{ijk}\,(S_{jk} + T_{jk}) = R^{ijk}\,S_{jk} + R^{ijk}\,T_{jk} \tag{1.2.20}$$

confirming its validity.

Application. The scalar-valued function

$$\rho = H^{ijkl}\, \alpha_{ij}\, \alpha_{kl}$$

has in symbolic notation the form

$$\rho = \boldsymbol{\alpha} : \mathbf{H} : \boldsymbol{\alpha} \quad \text{with} \quad \boldsymbol{\alpha} = \alpha_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j \ , \quad \mathbf{H} = H^{klmn}\, \mathbf{g}_k \otimes \mathbf{g}_l \otimes \mathbf{g}_m \otimes \mathbf{g}_n \ .$$

Partial derivatives of second-order tensors. To close this section we want to form the partial derivative of a second-order tensor $\mathbf{S} = S^{mn}\, \mathbf{g}_m \otimes \mathbf{g}_n$ with respect to the coordinate Θ^i. The corresponding result will be denoted as usual by $\mathbf{S},_i$. We first recall that the covariant derivative of the tensor components S^{mn} is defined by

$$S^{mn}|_i = S^{mn},_i + \Gamma^m_{ir}\, S^{rn} + \Gamma^n_{ir}\, S^{mr} \ , \tag{1.2.21}$$

in terms of the CHRISTOFFEL symbol $\Gamma^r_{ij} = \Gamma^r_{ji}$ of the second type, while the relation

$$\mathbf{g}_{i},_j = \frac{\partial \mathbf{g}_i}{\partial \Theta^j} = \Gamma^r_{ij}\, \mathbf{g}_r \tag{1.2.22}$$

holds for the partial derivatives of the basis $\mathbf{g}_i$. If we now construct the partial derivative $\mathbf{S},_i$ we receive

$$\begin{aligned} \mathbf{S},_i = \frac{\partial \mathbf{S}}{\partial \Theta^i} &= S^{mn},_i\, \mathbf{g}_m \otimes \mathbf{g}_n + S^{mn}\, \mathbf{g}_m,_i \otimes \mathbf{g}_n + S^{mn}\, \mathbf{g}_m \otimes \mathbf{g}_n,_i \\ &= (S^{mn},_i + \Gamma^m_{ir} S^{rn} + \Gamma^n_{ir} S^{mr})\, \mathbf{g}_m \otimes \mathbf{g}_n \\ &= S^{mn}|_i\, \mathbf{g}_m \otimes \mathbf{g}_n \ . \end{aligned} \tag{1.2.23}$$

Thus the covariant derivatives $S^{mn}|_i$ turn out to be the components of $\mathbf{S},_i$ with respect to the basis $\mathbf{g}_m \otimes \mathbf{g}_n$. If we define the covariant derivative of $\mathbf{g}_i$, similar to that of tensor components of type A_i, by

$$\mathbf{g}_i|_j = \mathbf{g}_i,_j - \Gamma^r_{ij}\, \mathbf{g}_r = \mathbf{0} \tag{1.2.24}$$

we see from (1.2.22) that the result is zero. Accordingly, covariant and partial derivatives of $\mathbf{S}$ are identical operations:

$$\mathbf{S},_i = \mathbf{S}|_i = S^{mn}|_i\, \mathbf{g}_m \otimes \mathbf{g}_n = S_{mn}|_i\, \mathbf{g}^m \otimes \mathbf{g}^n \ , \tag{1.2.25}$$

this result being valid for tensors of arbitrary order.

1.3 Special tensors

Tensors and vectors. In the following we shall introduce a number of special tensors playing an important role in tensor analysis and continuum mechanics. Most of the definitions will refer to second-order tensors which will be denoted by upper-case letters:

$$\mathbf{S} = S_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j \ , \quad \mathbf{T} = T_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j \ , \tag{1.3.1}$$

while vectors will be denoted as usual by lower-case letters, e.g.

$$\mathbf{u} = u_i \, \mathbf{g}^i \ , \quad \mathbf{w} = w_i \, \mathbf{g}^i \ . \tag{1.3.2}$$

Identity tensor. The *identity tensor* $\mathbf{I}$ is closely related to the base vectors $\mathbf{g}_i$ and is defined by

$$\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i = \mathbf{g}^i \otimes \mathbf{g}_i \ , \tag{1.3.3}$$

where, as usual, the summation rule is to be applied for repeated indices. The tensor $\mathbf{I}$ possesses according to (1.2.13) the following components

$$\mathbf{I} = g_{ij} \, \mathbf{g}^i \otimes \mathbf{g}^j = \delta_i^j \, \mathbf{g}^i \otimes \mathbf{g}_j = \delta_j^i \, \mathbf{g}_i \otimes \mathbf{g}^j = g^{ij} \, \mathbf{g}_i \otimes \mathbf{g}_j \ , \tag{1.3.4}$$

indicating $\mathbf{I}$ to be an invariant quantity associated with the metric tensor components $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$. Accordingly, $\mathbf{I}$ may be also called *metric tensor.* By means of the above definition (1.3.3) it can be shown that

$$\mathbf{I\,T} = \mathbf{T\,I} = \mathbf{T} \ . \tag{1.3.5}$$

This relation holds for arbitrary tensors $\mathbf{T}$, particularly also for vectors $\mathbf{u}$. In view of this property and (1.1.21) we receive:

$$\mathbf{I} : (\mathbf{c} \otimes \mathbf{d}) = \mathbf{c\,I\,d} = \mathbf{c} \cdot (\mathbf{I\,d}) = \mathbf{c} \cdot \mathbf{d} = c^i \, d_i \tag{1.3.6}$$

showing that $\mathbf{I}$ permits to express the scalar product of the vectors $\mathbf{c}$ and $\mathbf{d}$ in form of a double contraction.

If the basis $\mathbf{g}_i$ is changed into a new one $\bar{\mathbf{g}}_i$ associated with a new set of coordinates $\bar{\Theta}^i$

$$\mathbf{g}_i = \frac{\partial \bar{\Theta}^j}{\partial \Theta^i} \bar{\mathbf{g}}_j \quad , \qquad \mathbf{g}^i = \frac{\partial \Theta^i}{\partial \bar{\Theta}^k} \bar{\mathbf{g}}^k$$

then we receive from (1.3.3)

$$\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i = \frac{\partial \bar{\Theta}^j}{\partial \Theta^i} \frac{\partial \Theta^i}{\partial \bar{\Theta}^k} \bar{\mathbf{g}}_j \otimes \bar{\mathbf{g}}^k = \delta_k^j \, \bar{\mathbf{g}}_j \otimes \bar{\mathbf{g}}^k \tag{1.3.7}$$

which yields:

$$\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i = \bar{\mathbf{g}}_i \otimes \bar{\mathbf{g}}^i = \mathbf{i}_i \otimes \mathbf{i}^i \ . \tag{1.3.8}$$

Accordingly, the identity tensor $\mathbf{I}$ may be constructed from an arbitrary system of base vectors, particularly from those associated with orthogonal Cartesian coordinates, $\mathbf{i}_i$.

For later applications it is suitable to extend the definition (1.2.3) to an identity tensor of fourth-order

$$\overset{4}{\mathbf{I}} = \mathbf{I} \otimes \mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^k = g^{ij} \, g^{kl} \, \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes \mathbf{g}_l \tag{1.3.9}$$

which, by virtue of (1.2.15), satisfies the relations

$$\overset{4}{\mathbf{I}} : \mathbf{S} = \mathbf{S} : \overset{4}{\mathbf{I}} = (\operatorname{tr} \mathbf{S})\, \mathbf{I} \;, \quad \mathbf{S} : \overset{4}{\mathbf{I}} : \mathbf{S} = S_{i\cdot}^{\cdot i}\, S_{j\cdot}^{\cdot j} = (\operatorname{tr} \mathbf{S})^2 \;, \tag{1.3.10}$$

where $\mathbf{S}$ is an arbitrary second-order tensor and $\operatorname{tr} \mathbf{S}$ is an abbreviation for the double contraction $\operatorname{tr} \mathbf{S} = \mathbf{S} : \mathbf{I} = S_{i\cdot}^{\cdot i}$.

Inverse tensors. The *inverse tensor* $\mathbf{S}^{-1}$ of any second-order tensor $\mathbf{S}$ is defined by the equalities

$$\mathbf{S}\,\mathbf{S}^{-1} = \mathbf{S}^{-1}\,\mathbf{S} = \mathbf{I} \;, \tag{1.3.11}$$

$\mathbf{I}$ being the identity tensor which according to (1.3.5) satisfies the relation

$$\mathbf{I} = \mathbf{I}^{-1} \;. \tag{1.3.12}$$

The inverse tensor $\mathbf{S}^{-1}$ permits to solve any equation of the form

$$\mathbf{w} = \mathbf{S}\,\mathbf{v} \tag{1.3.13}$$

for $\mathbf{v}$ as

$$\mathbf{v} = \mathbf{S}^{-1}\,\mathbf{w} \;. \tag{1.3.14}$$

By means of the definition (1.3.11), the following identities can be derived for inverse tensors:

$$(\mathbf{S}^{-1})^{-1} = \mathbf{S} \;, \quad (\alpha\,\mathbf{S})^{-1} = \alpha^{-1}\,\mathbf{S}^{-1} \;, \quad (\mathbf{S}\,\mathbf{T})^{-1} = \mathbf{T}^{-1}\,\mathbf{S}^{-1} \;. \tag{1.3.15}$$

Application. As an example the last identity in (1.3.15) will be proved. For this purpose we start according to (1.3.11) from

$$(\mathbf{S}\,\mathbf{T})\,(\mathbf{S}\,\mathbf{T})^{-1} = \mathbf{I} \;,$$

which by multiplication with $\mathbf{T}^{-1}\,\mathbf{S}^{-1}$ from the left side delivers

$$\mathbf{T}^{-1}\,\mathbf{S}^{-1}\,\mathbf{S}\,\mathbf{T}\,(\mathbf{S}\,\mathbf{T})^{-1} = \mathbf{T}^{-1}\,\mathbf{S}^{-1} \;.$$

In view of (1.3.5) and (1.3.11) we then deduce that

$$(\mathbf{S}\,\mathbf{T})^{-1} = \mathbf{T}^{-1}\,\mathbf{S}^{-1} \;,$$

in accordance with (1.3.15).

Transposed tensors. A transposed tensor will be denoted by $(\ldots)^T$. By definition, any vector $\mathbf{u}$ is identical with its transposed $\mathbf{u}^T$. The same convention is also valid for a scalar α such that

$$\mathbf{u}^T = \mathbf{u} \;, \quad \alpha^T = \alpha \;. \tag{1.3.16}$$

The transposed tensor of a simple tensor

$$\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$$

is defined as a tensor obtained from $\mathbf{A}$ by interchanging the vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{A}^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a} \ . \tag{1.3.17}$$

This definition can be readily extended to a second-order tensor

$$\mathbf{S} = \mathrm{S}_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j \ .$$

If we replace the basis $\mathbf{g}^i \otimes \mathbf{g}^j$ according to (1.3.17) by its transposed form $\mathbf{g}^j \otimes \mathbf{g}^i$ the result

$$\mathbf{S}^{\mathrm{T}} = (\mathrm{S}^{\mathrm{T}})_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = \mathrm{S}_{ij}\, \mathbf{g}^j \otimes \mathbf{g}^i = \mathrm{S}_{ji}\, \mathbf{g}^i \otimes \mathbf{g}^j \tag{1.3.18}$$

is called the *transposed tensor* of $\mathbf{S}$ denoted by $\mathbf{S}^{\mathrm{T}}$. From (1.3.18) it follows that

$$(\mathrm{S}^{\mathrm{T}})_{ij} = \mathrm{S}_{ji} \ , \quad (\mathrm{S}^{\mathrm{T}})^{\cdot i}_{j} = \mathrm{S}_{j\cdot}^{\ i} \ , \quad (\mathrm{S}^{\mathrm{T}})_{i\cdot}^{\ j} = \mathrm{S}^{j}_{\cdot i} \ , \quad (\mathrm{S}^{\mathrm{T}})^{ij} = \mathrm{S}^{ji} \ . \tag{1.3.19}$$

The definition (1.3.18) of the transposed tensor $\mathbf{S}^{\mathrm{T}}$ may be equivalently replaced by the relation

$$\mathbf{u} \cdot (\mathbf{S}\, \mathbf{v}) = \mathbf{v} \cdot (\mathbf{S}^{\mathrm{T}}\, \mathbf{u}) = \mathbf{u}\, \mathbf{S}\, \mathbf{v} = \mathbf{v}\, \mathbf{S}^{\mathrm{T}}\, \mathbf{u} \tag{1.3.20}$$

holding for arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$. This can be easily proved by making use of component relations to obtain

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{S}\, \mathbf{v}) &= (\mathrm{u}_i\, \mathbf{g}^i) \cdot [(\mathrm{S}_{jk}\, \mathbf{g}^j \otimes \mathbf{g}^k)\, (\mathrm{v}_l\, \mathbf{g}^l)] \\
&= (\mathrm{u}_i\, \mathbf{g}^i) \cdot (\mathrm{S}_{jk}\, \mathrm{g}^{kl}\, \mathrm{v}_l\, \mathbf{g}^j) = \mathrm{u}_i\, \mathrm{g}^{ij}\, \mathrm{S}_{jk}\, \mathrm{g}^{kl}\, \mathrm{v}_l \\
&= \mathrm{u}_i\, \mathrm{S}^{il}\, \mathrm{v}_l = \mathbf{u}\, \mathbf{S}\, \mathbf{v} \ ,
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{S}^{\mathrm{T}}\, \mathbf{u}) &= (\mathrm{v}_l\, \mathbf{g}^l) \cdot [(\mathrm{S}^{\mathrm{T}})_{jk}\, \mathbf{g}^j \otimes \mathbf{g}^k)\, (\mathrm{u}_i\, \mathbf{g}^i)] \\
&= (\mathrm{v}_l\, \mathbf{g}^l) \cdot (\mathrm{S}^{\mathrm{T}})_{jk}\, (\mathrm{g}^{ki}\, \mathrm{u}_i\, \mathbf{g}^j) = \mathrm{v}_l\, \mathrm{g}^{lj}\, (\mathrm{S}^{\mathrm{T}})_{jk}\, \mathrm{g}^{ki}\, \mathrm{u}_i \\
&= \mathrm{v}_l\, (\mathrm{S}^{\mathrm{T}})^{li}\, \mathrm{u}_i = \mathrm{u}_i\, \mathrm{S}^{il}\, \mathrm{v}_l = \mathbf{v}\, \mathbf{S}^{\mathrm{T}}\, \mathbf{u} \ ,
\end{aligned}$$

which confirms in view of (1.3.19) the equality (1.3.20). By a similar procedure it can also be shown that

$$(\mathbf{T} + \mathbf{S})^{\mathrm{T}} = \mathbf{T}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}} \ , \tag{1.3.21}$$

$$(\mathbf{T}\, \mathbf{S})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\, \mathbf{T}^{\mathrm{T}} \ , \quad (\alpha\, \mathbf{S})^{\mathrm{T}} = \alpha\, \mathbf{S}^{\mathrm{T}} \ . \tag{1.3.22}$$

In view of (1.3.16) and (1.3.22) we have for any vector given as simple contraction $\mathbf{a} = \mathbf{S}\, \mathbf{u}$

$$\mathbf{a} = \mathbf{S}\, \mathbf{u} = \mathbf{a}^{\mathrm{T}} = \mathbf{u}\, \mathbf{S}^{\mathrm{T}} \ . \tag{1.3.23}$$

A further important identity is

$$\mathbf{A} : (\mathbf{B}\, \mathbf{C}) = (\mathbf{B}^{\mathrm{T}}\, \mathbf{A}) : \mathbf{C} = (\mathbf{A}\, \mathbf{C}^{\mathrm{T}}) : \mathbf{B} \ , \tag{1.3.24}$$