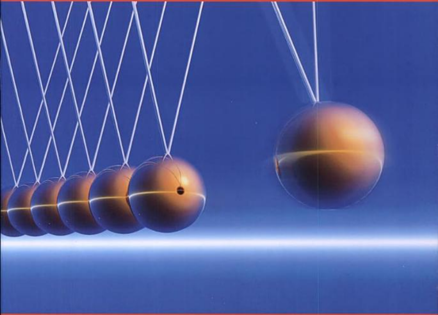
Introducing MECHANICS

BRIAN JEFFERSON
TONY BEADSWORTH



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Preface

Our intention with this book has been to produce a text covering the mechanics content of all the single-subject pure mathematics and mechanics specifications for A-level which will come into force in September 2000. We have not followed the syllabus of any one examination board, but have sought to develop the subject in such a way as to be accessible to all students.

We have tried to combine the best of the current approach, emphasising modelling and the 'real world' relevance of the subject, with some of the virtues of the more traditional texts. We have endeavoured to make modelling considerations the basis of the discussion of most topics,and, where appropriate, we have developed topics from the starting point of a practical problem or experiment. We have, however, not allowed this to compromise the need for a degree of mathematical rigour, and have included sufficient questions leading to solutions in algebraic form to satisfy those with a taste for such problems.

A special feature of the text is the reference to a number of spreadsheets, used to analyse the data from suggested experiments or to explore the implications of certain models. These can be downloaded from the Oxford University Press website (http://www.oup.co.uk/mechanics). While we do not claim any great sophistication for these, it is hoped they will be a useful resource in helping students gain a 'feel' for the subject.

The order in which topics have been covered is approximately that in which we choose to proceed in our own teaching. Naturally, this will not accord with everyone's approach, and the text contains a degree of cross-referencing to assist those wishing to 'dip in'.

The opening chapter of the book introduces the ideas of modelling and the modelling cycle, and emphasises the need to specify the assumptions made when developing a model and the importance of testing the predictions of that model against experimental data. In the next chapter, we develop the vector tools which underpin much of the subject. Chapter 3 explores the basic ideas of kinematics. This is followed by two chapters covering the concept of force and the all important Newton's laws. We then return to kinematics for a further three chapters, dealing with motion in two and three dimensions, the use of calculus, the concept of relative velocity.

Chapter 9 explores the problem of modelling friction, starting from a simple experimental approach. We then examine the concept of the moment of a force in Chapter 10, and consider the conditions necessary for equilibrium. Moments are then applied in the next chapter to finding centre of mass.

Chapters 12 and 13 are devoted to work, energy and power and to momentum respectively. The final six chapters deal with the 'harder' topics of frameworks, circular motion, elasticity and simple harmonic motion, together with a discussion of dimensional analysis and an introductory treatment of differential equations.

We anticipate that most students will use this book with the guidance of a teacher, but every effort has been made to make it readable and accessible to those using it for self-study or for revision. The exposition of topics proceeds by small steps and with a large number of worked examples to reinforce the ideas. The exercises are designed to give practice in the rote application of techniques, but also contain questions of a more testing nature. In addition, there are five sections containing a substantial selection of recent examination questions.

We are grateful to AEB, EDEXCEL, MEI, NEAB, NICCEA, OCR and WJEC for permission to use their questions. The answers provided for these questions are the sole responsibility of the authors.

We would like to express our thanks the Nigel Watts of King's School, Bruton, for the idea of 'modelling a skipper' used in the first chapter.

Thanks are also due to Rob Fielding and James Nicholson for checking the answers to the exercises and the examination questions. Finally, we owe an enormous debt of gratitude to John Day for his painstaking and detailed work in editing the book, and for his help and suggestions, which have contributed in no small measure to the final product.

Brian Jefferson
Tony Beadsworth
April 2000

1 Modelling

I cannot bring a world quite round, although I patch it as I can.
WALLACE STEVENS

A group of people on holiday with Explorer Tours proposes to drive directly across a stretch of desert from their present position A to a camp site at B. They consult their map of the region (scale 1 cm : 1 km), which clearly marks A and B, to decide how far they will need to drive.

They measure the straight line AB on their map with a ruler and find it to be 18.6 cm. They conclude that they will need to drive 18.6 km.

When they reach B, they check the distance they have travelled and find that it is 19.2 km.

Modelling reality

These people followed a process which is fundamental to the application of mathematics to real problems. They started with the real problem ...

'How far will we drive in going from A to B?'

set up a mathematical model ...

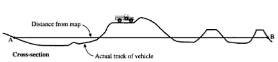
'The line AB on the map is a scale drawing of the journey.'

and from this model they obtained a solution to the problem. They then checked their solution against reality.

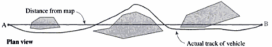
Simplifying assumptions

In setting up the model, the group made three simplifying assumptions.

- Using the map distance takes no account of hills and valleys. The model assumes that the journey is flat. That is, that any extra distance caused by hills is insignificant in relation to the length of the journey. The model would therefore tend to underestimate the actual distance driven.



- The model assumes that the journey is in an exact straight line. In practice, it is likely that there are rocks and other obstacles which they need to circumvent. So, again, the model is likely to produce an underestimate.



- The model assumes that the journey is so short that they can safely ignore any distortions caused by the fact that the line AB on the map is a flat projection of a journey taking place on the curved surface of the Earth. All map projections distort shapes and distances, the nature of the distortion depending on the particular method of projection used.

Comparison with reality: errors

Having solved their model, the group then did the journey, enabling them to compare their solution with the actual distance travelled. In making this comparison, they would need to be aware of sources of error, both in their prediction and in their measurement of reality.

- Their measurement of AB on the map is at best correct to one decimal place. This would place their predicted distance, D_P km, in the interval $18.55 \leqslant D_P < 18.65$. In addition, identifying their starting and finishing points on the map could only be an approximate affair, perhaps extending the error bounds to $18.5 \leqslant D_P < 18.7$.
- They found the actual distance using the odometer on their vehicle. This displayed 24 924.6 km at the start and 24 943.8 km at the end of their journey. These values are truncated to the nearest one decimal place below, which would put the start reading, S km, and the finish reading, F km, in the intervals $24\,924.6 \leqslant S < 24\,924.7$ and $24\,943.8 \leqslant F < 24\,943.9$ respectively.
- The minimum value of $(F - S)$ is therefore $24\,943.8 - 24\,924.7 = 19.1$ km and its maximum value is $24\,943.9 - 24\,924.6 = 19.3$ km. The actual distance, D_A km, would therefore have error bounds $19.1 \leqslant D_A < 19.3$. Even this assumes that the inevitable inaccuracy in the odometer mechanism was small enough to be insignificant over a short journey.

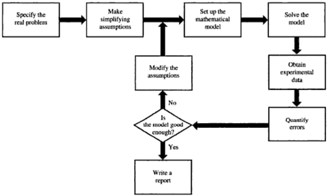
Was the model good enough?

Once the errors had been quantified as far as possible, the group would be able to decide whether their model was a sufficiently accurate representation of reality for their purpose. If not, they would need to re-examine the assumptions they made and modify the model. They might, for example, be able to obtain a larger-scale map and measure a route including detours around likely obstacles.

The modelling process

All applications of mathematics to real-world problems follow the same process as the one described above, which comprises the following eight steps:

1 **Specify the real problem** This should be a clear statement of the situation and should specify the results required in the solution.
2 **Make simplifying assumptions** All the factors which might affect the result should be considered and a decision made as to which should be taken into account in the model and which should be ignored. We may also make assumptions about the way in which certain variables are related. For example, we might decide to assume that air resistance is proportional to velocity.
3 **Set up the mathematical model** In the example given, this was a scale drawing, but it would more usually be a set of equations describing the behaviour of the simplified system.
4 **Solve the mathematical model** The equations should be solved to obtain the outcome which would result from the simplified system.
5 **Decide what really happens** This may involve setting up an experiment or obtaining data from published sources.
6 **Quantify the likely errors** There may be errors in the values used in the model and/or in the results obtained from the experiment. Error bounds should be established for all such values and the effects on the outcome should be quantified.
7 **Compare with reality** The results from the model should be compared with those obtained in reality to decide if the model provides a sufficiently accurate representation of the real situation. The errors mentioned in **6** need to be taken into account in this comparison.



8 Modify the model If the model does not give an adequate representation of the real situation, it is necessary to re-examine the assumptions on which it was based. A new model should then be set up to allow for the effect of one or more of the factors which had previously been ignored. The whole process should then be repeated, perhaps several times, until a sufficiently accurate model is obtained.

This process is summarised in the flowchart on page 3.

In this book, we concentrate on problems involving forces and the motion of objects, but the process of mathematical modelling is common to all situations in which mathematics is applied to real-world problems.

Another example of modelling

The problem is to model the motion of a person skipping.

We first need to state the precise questions which we wish to answer, for example:

- What is the relationship between the speed of the rope and the height of the jump?
- Are there limitations on these quantities for a given person and rope?

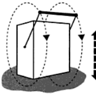
Our next task is to list all the factors which we think might have a bearing on the problem. This list can be as long and the factors as fanciful as you wish. It is better to include something a bit daft than to fail to take account of an important factor. Here is a possible list – you can probably think of several more items.

Length of rope
Mass of rope
Flexibility of rope
Thickness of rope
Whether the rope drags on the ground
Gravity
Height of person
Mass of person
Size of feet, length of arms and other physical proportions
Movement of arms and therefore the locus of rope
Speed of rope
Height of jump. Do we measure this as the movement of the person's centre of gravity or as the gap between the feet and the ground (bending of legs)?
Amount of time feet need to stay in contact with the ground in the jumping process
Air resistance
How 'bouncy' the ground is

Once we have our list, we must decide what assumptions to make.

For a first, simple model we might decide that a rope, which is curved and has mass all the way along, is too complex. It would be easier mathematically to replace it with a thin, rigid rod attached to two strings of negligible mass. In addition, it would be simpler if we supposed that the rope is being made to rotate at a constant speed in a circle around a fixed point in space, with the ground being a tangent to the circle.

The simplest way to model the person would be as a cuboid of uniformly dense material rising and falling without any change of shape. This would spend a fixed proportion of each cycle in contact with the ground and the rest moving vertically under gravity.



In this model, any resistance to the motion of the rope or the person would be ignored.

The important variables are the length, r, of the strings; the speed, v, of the rod around the circle; the height, h, of the jump; the time, t, from the start of the motion; and the proportion, p, of time spent in contact with the ground. On the assumption that the rope is at the bottom of the circle when the person is at the top of the jump, we could write equations connecting r, v, h, p and t. These equations would form the model and by manipulating them we could find solutions predicting the position of the rope and the person for any value of t.

There would be a lower limit on the rate of skipping because the value of v would have to be great enough to prevent the rope going slack at the top of the circle.

There would also be an upper limit because time would be needed for the person to get sufficiently high off the ground to allow the passage of the rope.

Our task would now be to observe people skipping, first to decide on a reasonable value for p and then to test out the predictions of our model about the relation between the height of jump and the speed of the rope.

It is unlikely that the model would be very good, so we would need to reassess our assumptions. Observing skippers would help us to decide which assumptions to modify. We would continue to refine our model and test against observation until we regarded the predictions as sufficiently accurate.

Conventional terms

When stating problems in mathematics, we often use terms which imply that certain assumptions are being made. For example, you will see questions referring to a string as *light*. This would indicate that the mass of the string is sufficiently small for it to be ignored.

The common terms are given in the table below.

Term	Applies to	What is disregarded
Inextensible	Strings, rods	Stretching
Light	Strings, springs, rods	Mass
Particle	Object of negligible size	Rotational motion
Rigid	Rods	Bending
Small	Object of negligible size	Rotational motion
Smooth	Surfaces, pulleys	Friction

Exercise 1

1 Do you think it would be reasonable to disregard air resistance in the following situations?

a) A marble dropped from an upstairs window.
b) A table tennis ball dropped from an upstairs window.
c) A marble dropped from an aircraft at 2000 metres altitude.
d) A shot being putt.
e) A rocket firework being set off.
f) A child on a swing.
g) A person walking.
h) A person cycling.

2 Do you think it would be reasonable to disregard friction in the following situations?

a) Skiing downhill.
b) A child going down a slide.
c) Raising an object on a rope passing over a tree branch.
d) Raising an object on a rope passing over a pulley.
e) A car being driven in a straight line.
f) A car being driven round a curve.

3 In the sport of bungee jumping, participants jump from a platform with an elastic rope attached to their ankles. The other end of the rope is attached to the platform. Participants free fall until the elastic stretches. The tension built up in the elastic slows them down and eventually brings them to a temporary stop. Often the jump takes place over water and the participants have the choice of whether to come to a stop before they hit the water, whether to get their hair wet or whether to plunge into the water to a depth chosen by them. The problem is to work out the correct length of rope to satisfy their desires.

In modelling this problem the following list of factors was drawn up. Separate them into three lists:

A Those which can be totally ignored in forming a mathematical model.
B Those which cannot be ignored but which you think would be too difficult to include in an initial model.
C Those which should probably be included in an initial model.

In each case, while at this stage you do not have enough knowledge to answer this question with 100% confidence, try to justify your inclusion of each item in its list.

a) The weight of the person.
b) The height of the person.
c) The height of the platform.
d) The elasticity of the bungee rope.
e) The number of ropes used.
f) The accuracy with which the measurements can be made.
g) The weight of the bungee rope.
h) The weather conditions.
i) The depth of the water.
j) The style of jumping.
k) Air resistance.
l) The clothing worn.
m) The maximum stress the body can take.
n) The way the bungee rope deforms when it is stretched.
o) How fast the water is flowing.
p) How fast the person wants to be moving when he/she hits the water.
q) Whether there is a cross wind.
r) How the rope is tied to the ankles.
s) Any more you can think of.

4 For each of the following situations, make a list of the factors which you think might have a bearing on the outcome.

a) The amount of water falling on a person crossing an open space in the rain.
b) The motion of a boat crossing a river.
c) A tennis player serving.
d) A toy car free-wheeling from rest down a slope.
e) A child swinging on a rope tied to a tree branch.

2 Vectors

Lord Ronald . . . flung himself upon his horse and rode madly off in all directions.
STEPHEN LEACOCK

When modelling physical systems, we use a number of quantities, such as force, displacement, velocity, acceleration and momentum, which share a common property: namely, all of these quantities can be specified completely only by stating **both** their **magnitude** (size) and their **direction**. Such quantities are called **vectors**.

A **vector quantity** is one which has both **magnitude** and **direction**.

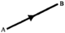
We also use other quantities, such as distance, speed, work and power, which are completely specified by their magnitude. Such quantities are called **scalars**.

A **scalar quantity** is one which has **only magnitude**.

Because of this shared vector property, the mathematical techniques used for combining and manipulating displacements work equally well when we wish to combine and manipulate forces or velocities. We therefore need to spend some time becoming familiar with the language and mathematics of vectors.

Notation

The simplest vector quantity to illustrate is a displacement, or translation, for a given distance in a given direction. This can be represented by a directed line segment.



The line segment shown in the diagram represents a translation from A to B. To show that it is a translation rather than just the distance AB, an arrow is put over the pair of letters to give $\overrightarrow{AB}$. This convention is the more widely used, particularly by the examination boards. (The other way to represent a directed line segment is to print its pair of letters in a bold face to give, in our example, **AB**.)

An alternative way of labelling vectors is to use a single letter in bold type, such as **a**. This would be handwritten as $\underset{\sim}{a}$.

Magnitude

The magnitude of the vector $\overrightarrow{AB}$ is shown as AB or $|\overrightarrow{AB}|$.

The magnitude of the vector **a** is shown as $|\mathbf{a}|$ or a.

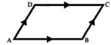
Unit vector

A vector with a magnitude of 1 unit is called a **unit vector.** The unit vector in the direction of a vector **a** is usually labelled $\hat{\mathbf{a}}$ (often referred to as 'a hat').

Properties of vectors

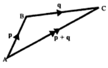
Equality of vectors

Vectors are equal if they have the same magnitude and direction. For example, in the parallelogram shown on the right, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.



Addition of vectors

In the triangle on the right, we can see that if we combine the translations $\overrightarrow{AB}$ and $\overrightarrow{BC}$, the effect would be the same as the single translation $\overrightarrow{AC}$.



We say that $\overrightarrow{AC}$ is the **vector sum** of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and write

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$\overrightarrow{AC}$ is called the **resultant** of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(Note the use of a double arrowhead in the diagram to signify that the vector is a resultant.)

You should be clear that this does **not** mean that the lengths AB + BC = AC. Think of the + symbol as meaning 'followed by', so $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ means

translation $\overrightarrow{AB}$ followed by translation $\overrightarrow{BC}$ is equivalent to translation $\overrightarrow{AC}$

The justification for using the + symbol will be clear when we consider vectors in component form.

Zero vector

If we were to combine the displacements $\overrightarrow{AB}$ and $\overrightarrow{BA}$, the resultant would be a vector with zero magnitude (its direction would be undefined). We call this the **zero vector** and write it as **0** (handwritten $\underset{\sim}{0}$).

Negative vectors

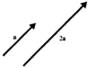
As $\overrightarrow{AB} + \overrightarrow{BA} = \mathbf{0}$, it is reasonable to write $\overrightarrow{BA} = -\overrightarrow{AB}$.

The translations $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are the exact opposites of each other.

In general, the vector $-\mathbf{a}$ has the same magnitude as $\mathbf{a}$ but the opposite direction.

Multiplying by a scalar

If the translation $\mathbf{a}$ is applied twice, the effect is a translation twice as far in the same direction:



$$\mathbf{a} + \mathbf{a} = 2\mathbf{a}$$

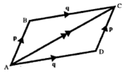
$$|2\mathbf{a}| = 2\,|\mathbf{a}|$$

In general, $k\mathbf{a}$ is a vector parallel to $\mathbf{a}$ and with magnitude $k\,|\mathbf{a}|$:

$$|k\mathbf{a}| = k|\mathbf{a}|$$

Commutativity

Translation $\mathbf{p}$ followed by translation $\mathbf{q}$ has the same resultant as $\mathbf{q}$ followed by $\mathbf{p}$. In the diagram



$$\mathbf{p} + \mathbf{q} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

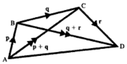
$$\mathbf{q} + \mathbf{p} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$$

$$\Rightarrow \quad \mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$$

That is, vector addition is **commutative.**

Associativity

If we add several vectors, the order in which we bracket them does **not affect** the resultant. In the diagram:



$$(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$$

$$\mathbf{p} + (\mathbf{q} + \mathbf{r}) = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$$

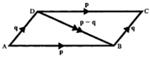
$$\Rightarrow \quad (\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$$

That is, vector addition is **associative.**

Subtraction

Subtracting the vector $\overrightarrow{BC}$ is equivalent to adding $-\overrightarrow{BC}$, or $\overrightarrow{CB}$. In the diagram, $\mathbf{p} = \overrightarrow{AB}$ and $\mathbf{q} = \overrightarrow{AD}$, so:

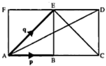
$$\begin{aligned}\mathbf{p} - \mathbf{q} &= \overrightarrow{AB} - \overrightarrow{AD}\\ &= \overrightarrow{AB} + \overrightarrow{DA}\\ &= \overrightarrow{DA} + \overrightarrow{AB}\\ &= \overrightarrow{DB}\end{aligned}$$



Example 1 In the diagram, ABEF and BCDE are squares. Vector $\overrightarrow{AB} = \mathbf{p}$ and vector $\overrightarrow{AE} = \mathbf{q}$. Find **a)** $\overrightarrow{AC}$ **b)** $\overrightarrow{AD}$ **c)** $\overrightarrow{AF}$ **d)** $\overrightarrow{EC}$

SOLUTION

a) $\overrightarrow{AC} = 2\overrightarrow{AB} \Rightarrow \overrightarrow{AC} = 2\mathbf{p}$



b) $\overrightarrow{AD} = \overrightarrow{AE} + \overrightarrow{ED} = \mathbf{q} + \overrightarrow{ED}$

But $\overrightarrow{ED} = \overrightarrow{AB} = \mathbf{p}$

$\Rightarrow \overrightarrow{AD} = \mathbf{q} + \mathbf{p}$

Note Any route from A to D gives the required result. For example, we could have said

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$$

As $\overrightarrow{BD} = \overrightarrow{AE} = \mathbf{q}$, this gives

$$\overrightarrow{AD} = \mathbf{p} + \mathbf{q}$$

c) $\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF} = \mathbf{q} + \overrightarrow{EF}$

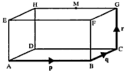
But $\overrightarrow{EF} = \overrightarrow{BA} = -\mathbf{p}$

$\Rightarrow \overrightarrow{AF} = \mathbf{q} - \mathbf{p}$

d) $\overrightarrow{EC} = \overrightarrow{EA} + \overrightarrow{AC} = -\mathbf{q} + 2\mathbf{p}$

Example 2 The diagram shows a cuboid ABCDEFGH, with $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CG}$ corresponding to the vectors **p**, **q** and **r**, as shown. M is the mid-point of GH. Find, in terms of **p**, **q** and **r**, the following vectors:

a) $\overrightarrow{AC}$ **b)** $\overrightarrow{DF}$ **c)** $\overrightarrow{BM}$



SOLUTION

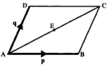
a) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{p} + \mathbf{q}$

b) $\overrightarrow{DF} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BF} = \mathbf{p} - \mathbf{q} + \mathbf{r}$

c) $\overrightarrow{BM} = \overrightarrow{BC} + \overrightarrow{CG} + \overrightarrow{GM} = \mathbf{q} + \mathbf{r} - \frac{1}{2}\mathbf{p}$

Example 3 ABCD is a parallelogram. E is the mid-point of AC. Vector $\overrightarrow{AB} = \mathbf{p}$ and vector $\overrightarrow{AD} = \mathbf{q}$. Find **a)** $\overrightarrow{BE}$ **b)** $\overrightarrow{BD}$

What can be deduced from the result?



SOLUTION

a) First notice that

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{p} + \mathbf{q}$$

and

$$\overrightarrow{AE} = \tfrac{1}{2}\overrightarrow{AC} = \tfrac{1}{2}(\mathbf{p} + \mathbf{q})$$

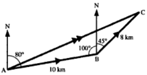
Therefore,

$$\begin{aligned}\overrightarrow{BE} &= \overrightarrow{BA} + \overrightarrow{AE} \\ &= -\mathbf{p} + \tfrac{1}{2}(\mathbf{p} + \mathbf{q}) \\ &= \tfrac{1}{2}(\mathbf{q} - \mathbf{p})\end{aligned}$$

b) $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \mathbf{q} - \mathbf{p}$

We can see from this that $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BD}$. This means that BE is half the length of BD and BED is a straight line. That is, E is the mid-point of BD. This proves that the diagonals of a parallelogram bisect each other. (Many standard geometrical theorems can be proved by vector methods in this way.)

Example 4 An expedition in the Sahara travels 10 km on a bearing of 080° and then 8 km on a bearing of 045°. What is the expedition's final position in relation to its starting point?



SOLUTION

The resultant of the two stages of the journey is the vector $\overrightarrow{AC}$ in the diagram.

In triangle ABC, we have AB = 10 km, BC = 8 km and $A\hat{B}C = 145°$.

By the cosine rule:

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2 \times AB \times BC \times \cos A\hat{B}C \\ &= 10^2 + 8^2 - 2 \times 10 \times 8 \times \cos 145^\circ \\ &= 295.08 \end{aligned}$$

$$\Rightarrow \quad AC = 17.18\,\text{km}$$

By the sine rule:

$$\frac{AC}{\sin A\hat{B}C} = \frac{BC}{\sin B\hat{A}C}$$

$$\Rightarrow \quad \frac{17.18}{\sin 145^\circ} = \frac{8}{\sin B\hat{A}C}$$

$$\Rightarrow \quad \sin B\hat{A}C = \frac{8 \sin 145^\circ}{17.18} = 0.2671$$

$$\Rightarrow \quad B\hat{A}C = 15.5^\circ$$

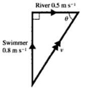
So, $\overrightarrow{AC}$ is a displacement of 17.18 km on a bearing of $080^\circ - 015.5^\circ = 064.5^\circ$.

Example 5 A swimmer, who can swim at $0.8\,\text{m}\,\text{s}^{-1}$ in still water, wishes to cross a river flowing at $0.5\,\text{m}\,\text{s}^{-1}$.

a) If she aims straight across the river, what will be her actual velocity?
b) If she wishes to travel straight across, in what direction should she aim and what will be her actual speed?

SOLUTION

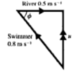
We need to make the simplifying assumption that the water flows at a uniform speed at all points on the crossing. We can then represent the velocities by the vector diagrams on the right.



a) The swimmer's actual speed is v, so $v = \sqrt{0.8^2 + 0.5^2} = 0.943\,\text{m}\,\text{s}^{-1}$
Her direction θ is given by

$$\tan\theta = \frac{0.8}{0.5} \quad \Rightarrow \quad \theta = 58^\circ$$

So the swimmer travels at $0.943\,\text{m}\,\text{s}^{-1}$ at an angle of 58° to the direction of the river.



b) The direction of the swimmer's aim is given by

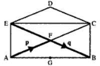
$$\cos\phi = \frac{0.5}{0.8} \quad \Rightarrow \quad \phi = 51.3^\circ$$

Her actual speed is $u = \sqrt{0.8^2 - 0.5^2} = 0.625\,\mathrm{m\,s^{-1}}$.

So, she should aim upstream at 51.3° to the bank. She will then travel straight across the river at $0.625\,\mathrm{m\,s^{-1}}$.

Exercise 2A

1 ABCE is a rectangle. CDEF is a rhombus. G is the mid-point of AB. $\overrightarrow{AF} = \mathbf{p}$ and $\overrightarrow{EB} = \mathbf{q}$.

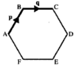


a) Find in terms of **p** and **q**:

i) $\overrightarrow{AB}$ **ii)** $\overrightarrow{CB}$ **iii)** $\overrightarrow{DB}$

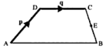
b) Show that $\overrightarrow{EB} + \overrightarrow{CA} = 2\overrightarrow{DF}$.

2 The diagram shows a regular hexagon ABCDEF with $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$. Find in terms of **p** and **q**:



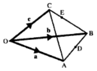
a) $\overrightarrow{AD}$ **b)** $\overrightarrow{AC}$ **c)** $\overrightarrow{CE}$ **d)** $\overrightarrow{BE}$ **e)** $\overrightarrow{EA}$

3 The diagram shows a trapezium ABCD with AB parallel to DC and twice as long. E is the mid-point of BC. $\overrightarrow{AD} = \mathbf{p}$ and $\overrightarrow{DC} = \mathbf{q}$. Find in terms of **p** and **q**:



a) $\overrightarrow{AB}$ **b)** $\overrightarrow{AC}$ **c)** $\overrightarrow{CD}$ **d)** $\overrightarrow{DB}$ **e)** $\overrightarrow{AE}$ **f)** $\overrightarrow{ED}$

4 The diagram shows a tetrahedron OABC with $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. D is the mid-point of AB and E is on BC so that the ratio BE : EC = 2 : 1. Find in terms of **a**, **b** and **c**:

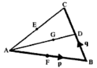


a) $\overrightarrow{AC}$ **b)** $\overrightarrow{AB}$ **c)** $\overrightarrow{AD}$ **d)** $\overrightarrow{BC}$

e) $\overrightarrow{BE}$ **f)** $\overrightarrow{OE}$ **g)** $\overrightarrow{DE}$

5 Use vector methods to show that the line joining the mid-points of two sides of a triangle is parallel to the third side and half its length.

6 The diagram shows triangle ABC with D, E and F the mid-points of BC, AC and AB respectively. G is the point on AD such that the ratio AG : GD = 2 : 1. Vector $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$.



a) Find in terms of **p** and **q**:

i) $\overrightarrow{DB}$ **ii)** $\overrightarrow{DA}$ **iii)** $\overrightarrow{BG}$ **iv)** $\overrightarrow{GE}$

Explain what your results indicate about the points B, G and E.

b) Prove the equivalent result for points C, G and F.